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THE CALCULUS OF CLOVERS

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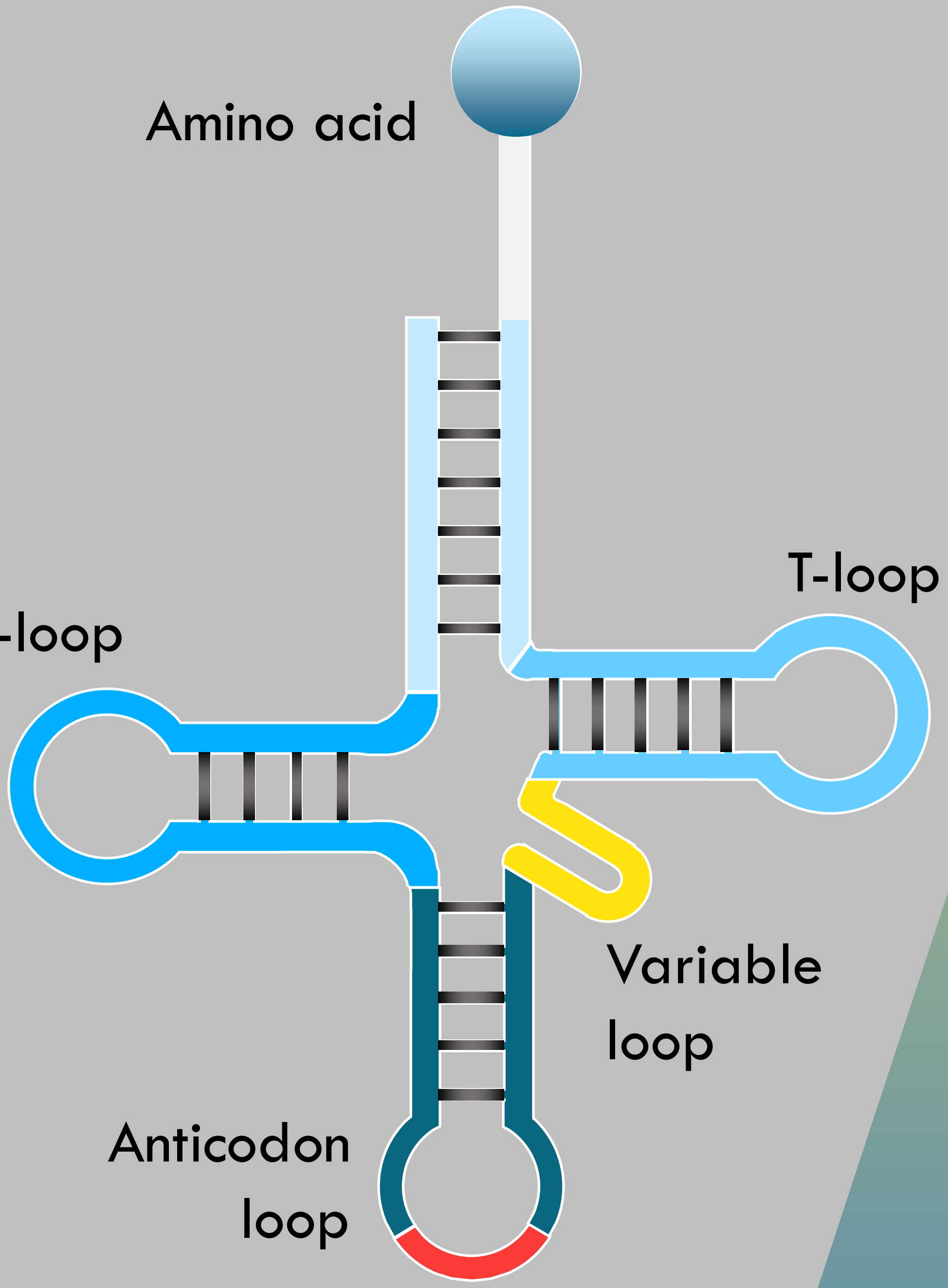
The clover-leaf is a fundamental shape that manifests often in nature.



courses.lumenlearning.com

Transfer ribonucleic acid (tRNA) is a type of RNA molecule that helps decode a messenger RNA (mRNA) sequence into a protein and has a built in Cloverleaf structure.

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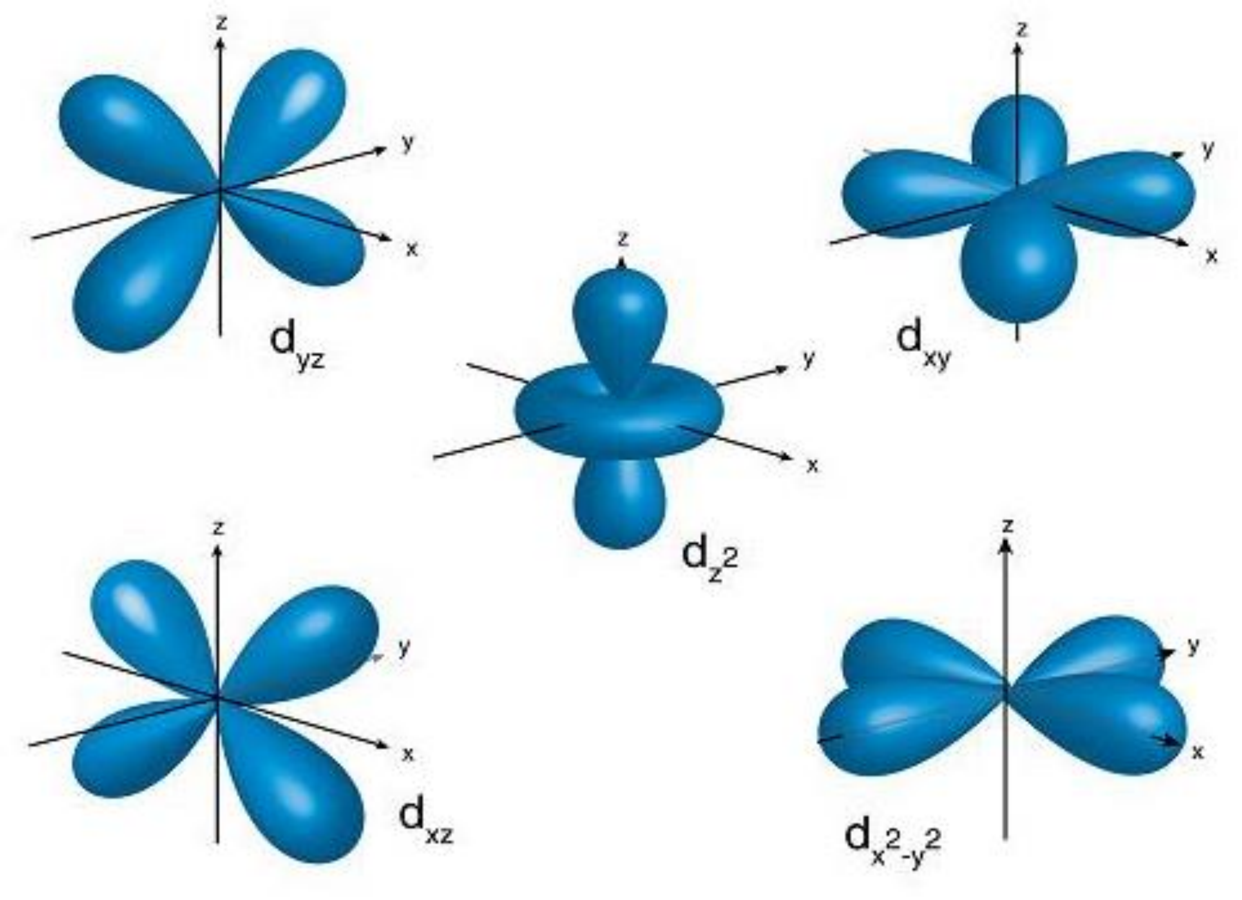
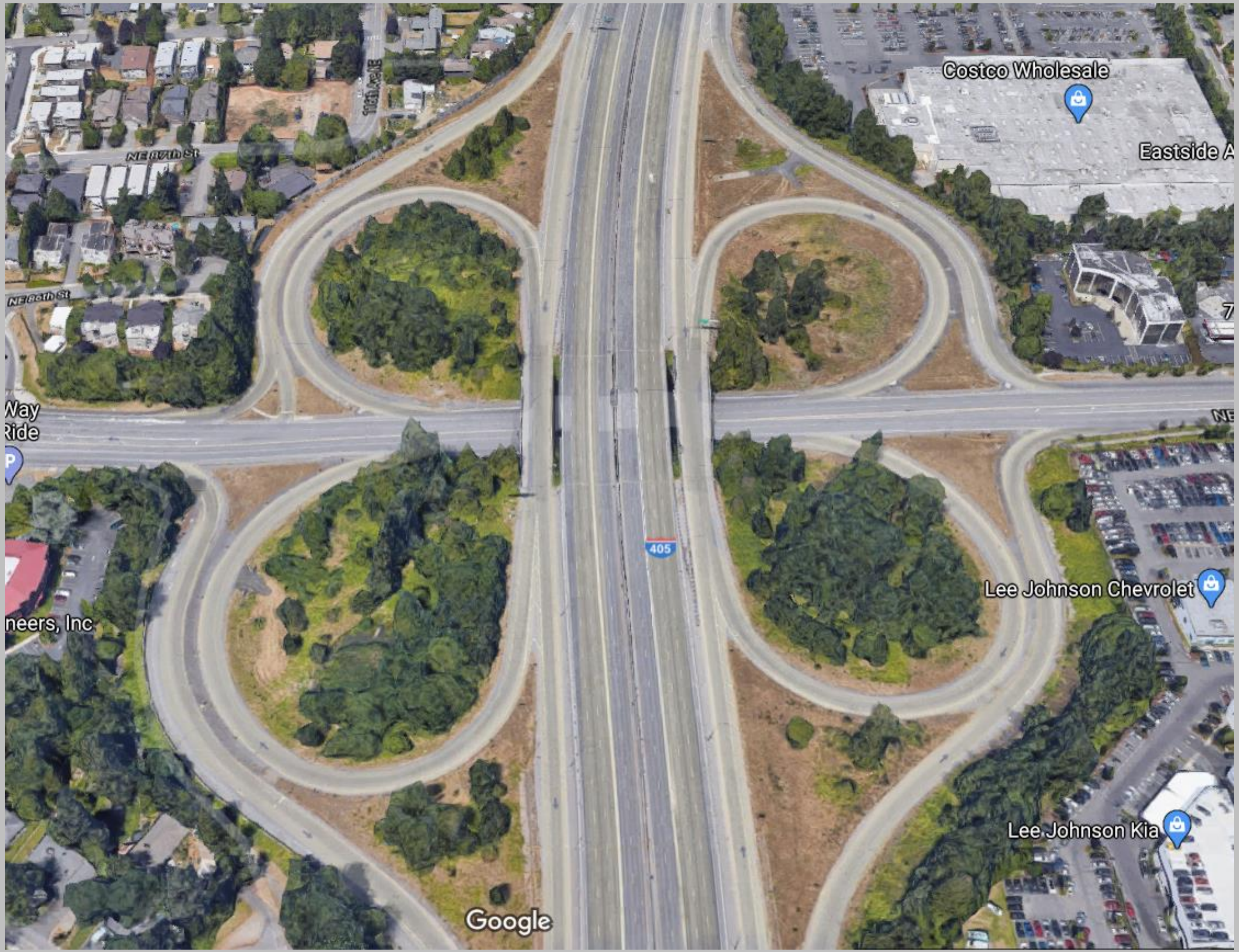


Socratic.org

The atomic d-orbitals have a three-dimensional clover leaf shape.



Google.com/maps

We have a cloverleaf interchange at 85th Street in Kirkland

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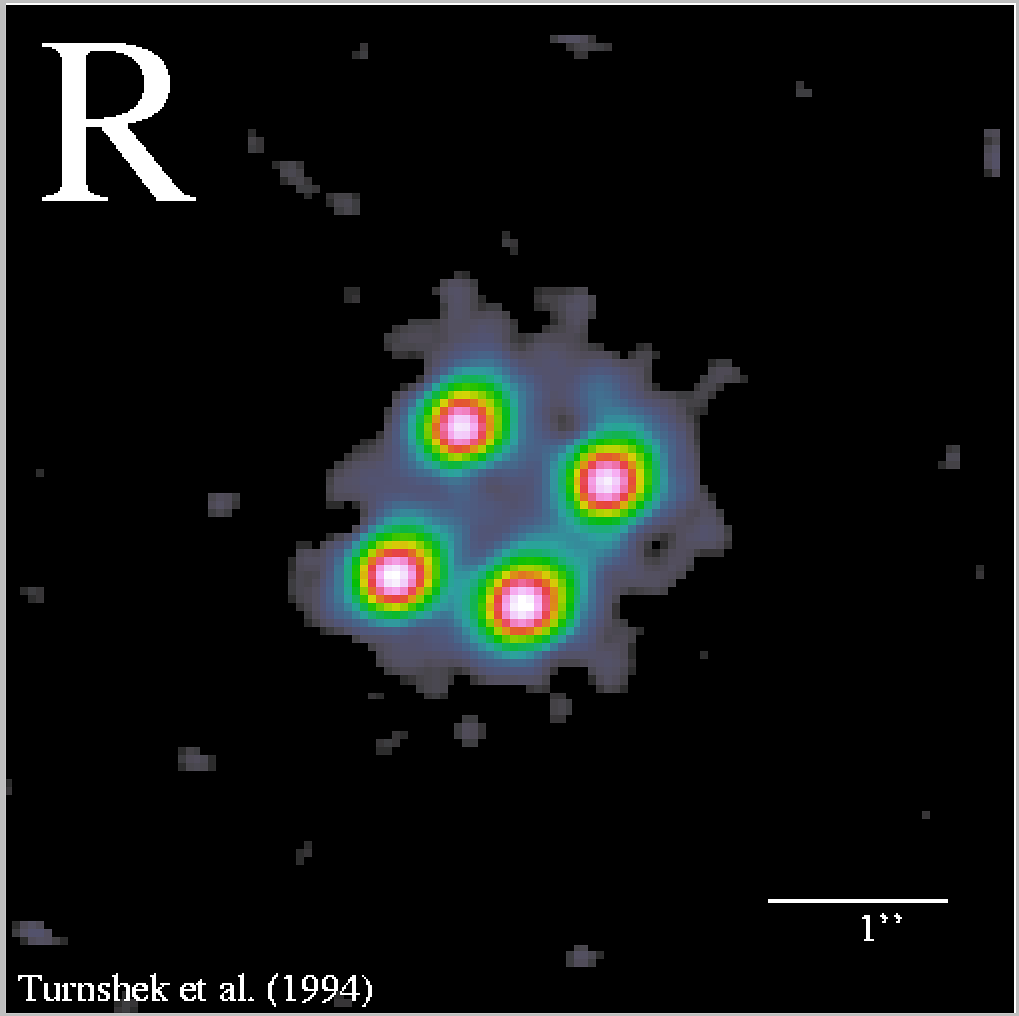

chandra.harvard.edu

The Chandra x-ray observatory discovered exciting findings of clover-leaf quasars that provides evidence of large-scale star formation in the early universe.

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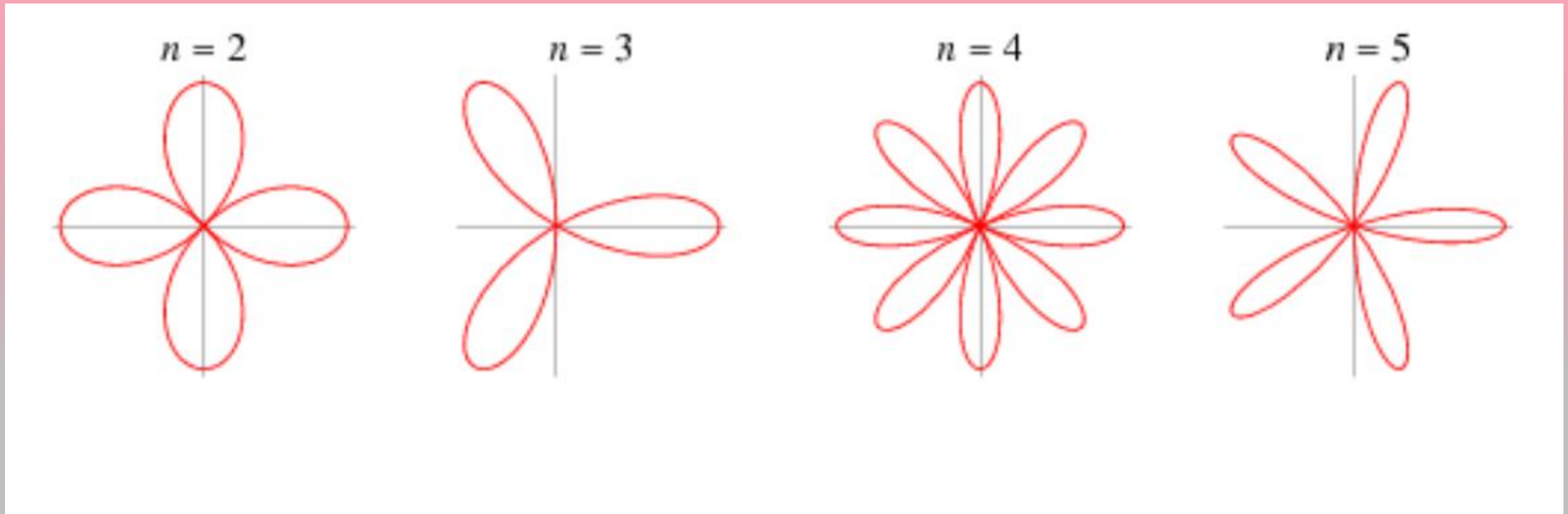
- Here, we developed Mathematical models.

- We have used multiple integration involving double and triple integrals in polar and cylindrical coordinates to calculate the areas and volumes of these shapes.

MATHEMATICAL MODELS

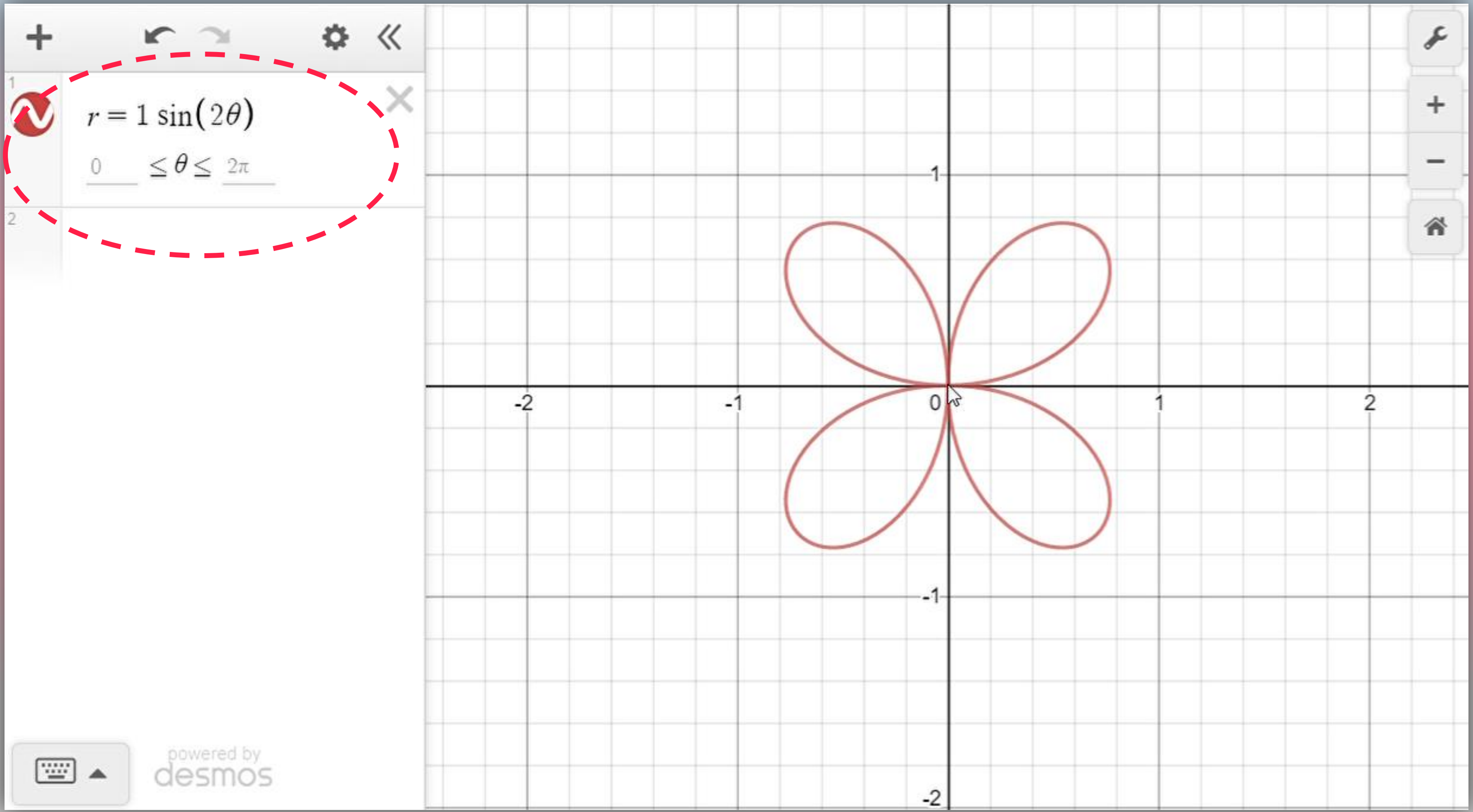
A curve which has the shape of a petalled flower. This curve was named rhodonea by the Italian mathematician Guido Grandi between 1723 and 1728 because it resembles a rose (MacTutor Archive). The polar equation of the rose is $r = a \cdot \sin(n\theta)$.

If the n is odd, the rose is n-petalled. If n is even, the rose is 2n-petalled.



mathworld.wolfram.com/Rose.html

MATHEMATICAL MODELS



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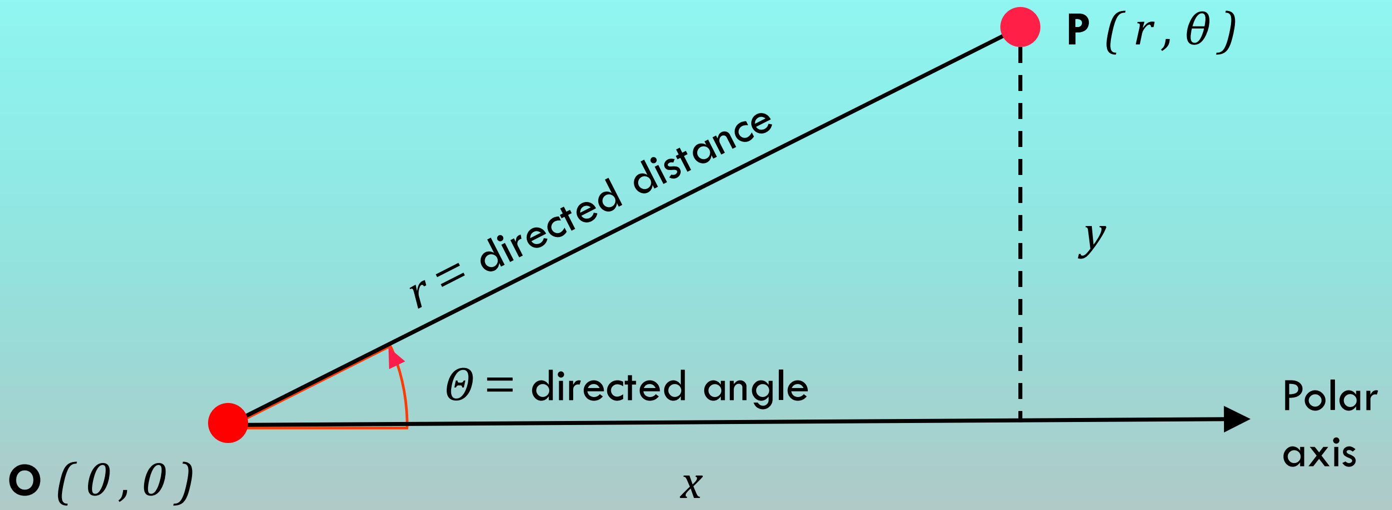
Polar Coordinates

The **polar coordinate system** is formed by fixing a point $\boldsymbol{O}$ which is the **pole** (or **origin**).

The **polar axis** is the ray constructed from $\boldsymbol{O}$.

Each **point** ***P*** in the plane can be assigned a **polar coordinate** (r, θ).



r is the directed distance from $\boldsymbol{O}$ to $\boldsymbol{P}$.

$\boldsymbol{\theta}$ is the directed angle (counterclockwise) from the polar axis to $\overline{\boldsymbol{OP}}$.

$$x = r \cdot \cos(\theta)$$

$$y = r \cdot \sin(\theta)$$

$$\tan(\theta) = \frac{y}{x}$$

$$r^2 = x^2 + y^2$$

To transform from Cartesian to polar coordinates, we need to evaluate the determinant of the **Jacobian** matrix.

$$x = r \cdot \cos(\theta)$$

$$y = r \cdot \sin(\theta)$$

$$\mathbf{J_F}(r, \theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r \cdot \sin(\theta) \\ \sin(\theta) & r \cdot \cos(\theta) \end{bmatrix} = r$$

The Jacobian determinant is equal to *r*. This can be used to transform integrals between the two coordinate systems

To transform from 3D Cartesian to Cylindrical Coordinates, we have to use the same procedure.

$x = r \cdot \cos(\theta)$

$y = r \cdot \sin(\theta)$

$z = z$

$$J_F(r, \theta, z) = \begin{vmatrix} cos\theta & -rsin\theta & 0 \\ sin\theta & rcos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Area elements (dA)

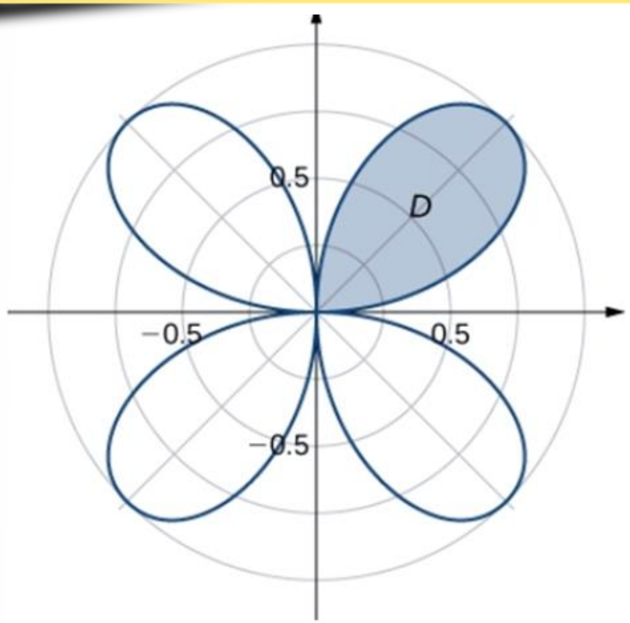
Cartesian	$dxdy = dydx$
Polar	$rdrd\theta$

Volume elements (dV)

Cartesian	$dxdydz=dydxdz=dzdxdy$
Cylindrical	$rdzdrd\theta$

OpenStax Calculus Volume 3, Problem 155

Find the total area of the region enclosed by the four-leaved rose r = sin2θ.



The region bounded by the first branch is

$$\left\{(r,\theta)\middle|\ 0 \le r \le \sin(2\theta),\ \ 0 \le \theta \le \frac{\pi}{2}\right\}$$

$$A = 4\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{\sin 2\theta} d\theta$$

$$= 4\int_0^{\pi/2}\frac{\left(\sin^2(2\theta)-0\right)}{2}d\theta$$

$$= 2\int_0^{\pi/2}\frac{\left(1-\cos(4\theta)\right)}{2}d\theta$$

$$= \left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0 - (0-0)$$

$$\boxed{= \frac{\pi}{2}\ \text{units}^2}$$

The clover shaped swimming pool has equation
$r=10\sin(2\Theta)$
If the height in ft is given by

$z=f(x,y)= x+y$ ft, then setup integrals to solve

(i) Graph the swimming pool using DESMOS polar calculator
(ii) Find the volume of water in the pool,
(iv) Find the average height/depth of water in the pool.

Use only the clover leaf portion of the pool shown.

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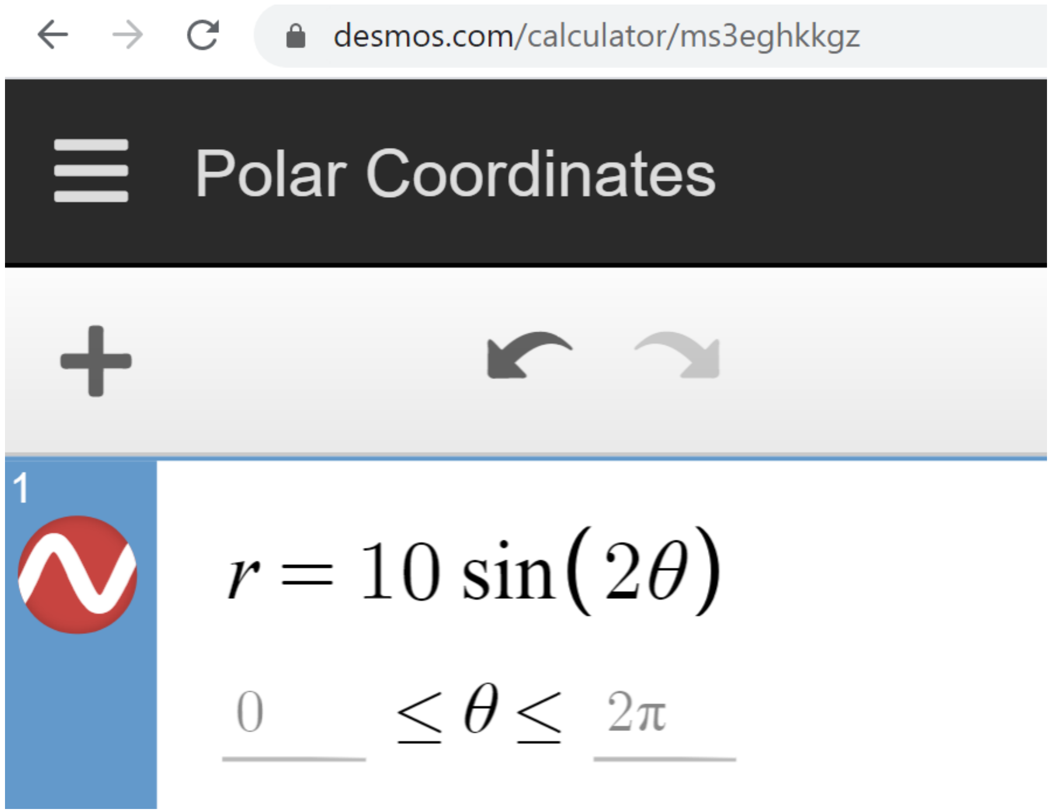
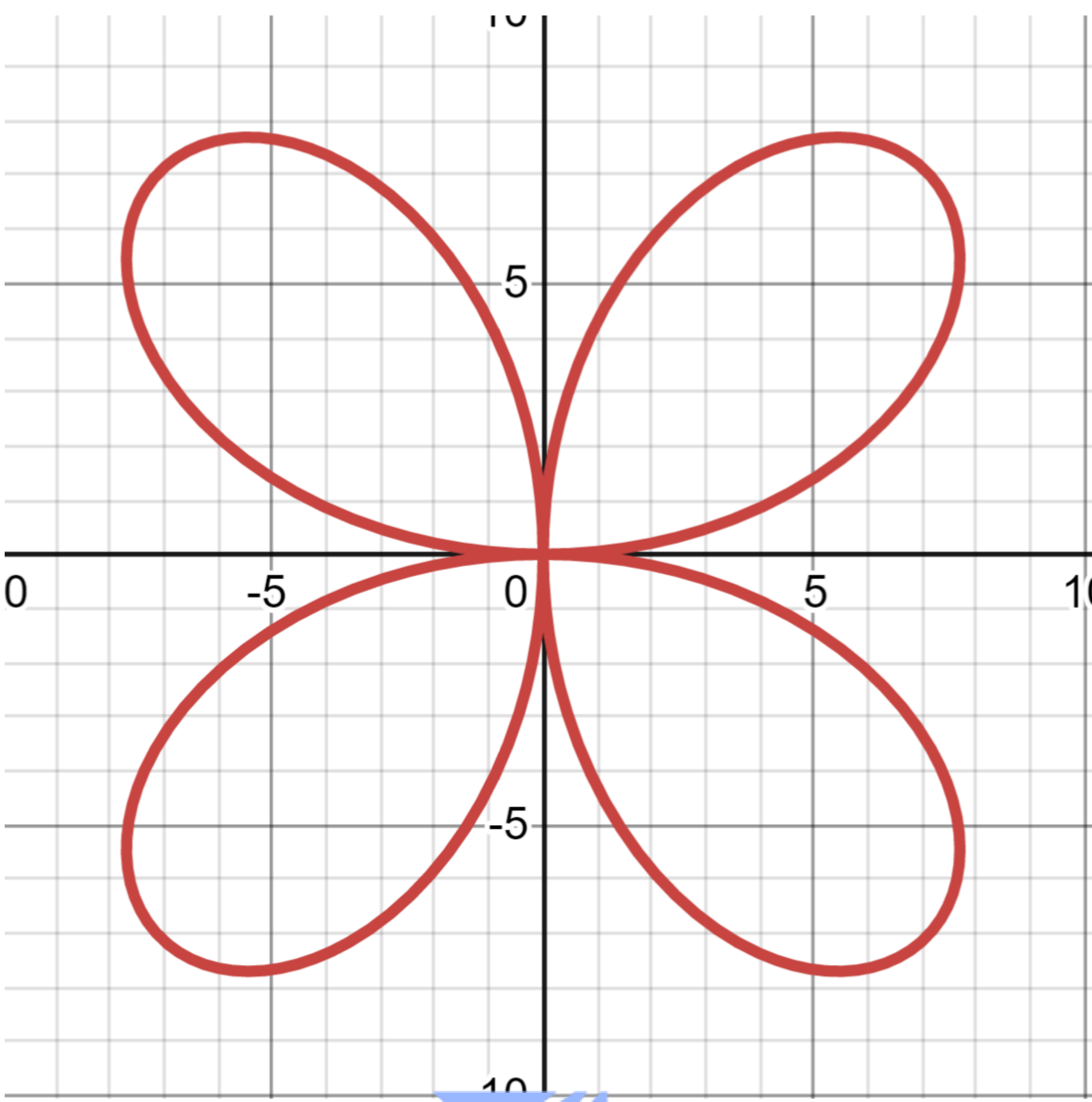
- Graph using DESMOS polar calculator





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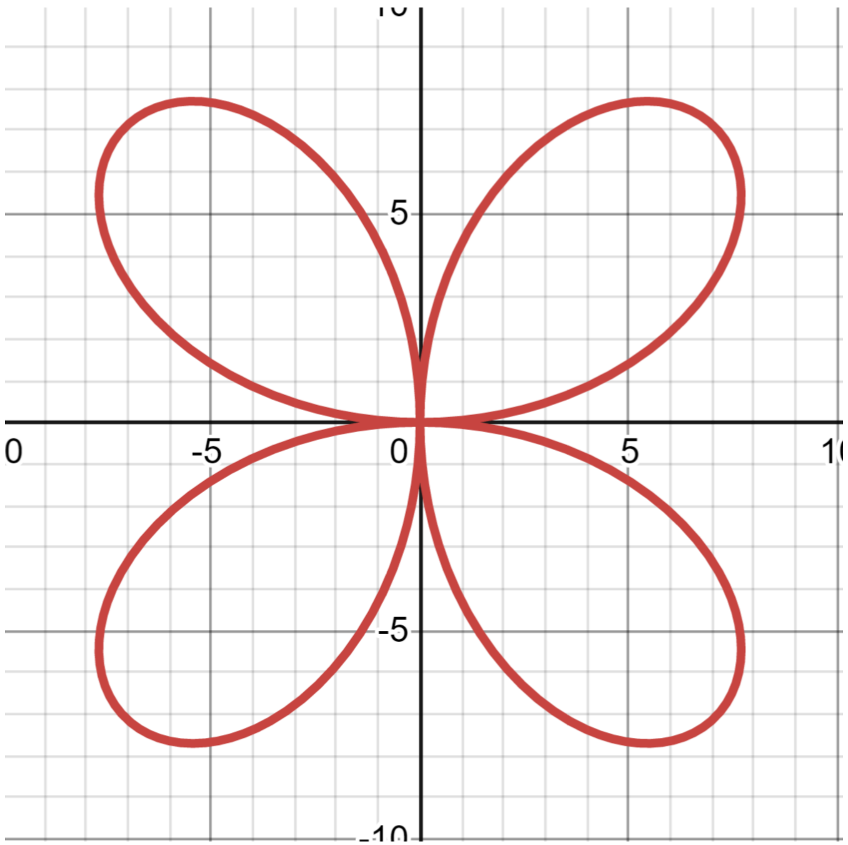
$$V = \int_{\theta 1}^{\theta 2} \int_{r1}^{r2} \int_{z1}^{z2} rdzdrd\theta$$



$$x = rcos\theta \quad y = rsin\theta \quad z = x + y$$

$$z = rcos\theta + rsin\theta$$

$$r = 10rsin(2\theta)$$

$$V = 4\int_0^{\frac{\pi}{2}} \int_0^{10sin2\theta} \int_0^{rcos\theta + rsin\theta} rdzdrd\theta$$

$$V = 1219.05\,ft^3$$

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$$V = 4\int_0^{\pi/2}\int_0^{10\sin(2\theta)}\int_0^{r(\cos(\theta)+\sin(\theta))} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{10\sin(2\theta)} rz\Big|_0^{r(\cos(\theta)+\sin(\theta))} dz\,d\theta \quad = 4\int_0^{\pi/2}\int_0^{10\sin(2\theta)} r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \frac{r^3}{3}(\cos\theta+\sin\theta)\Big|_0^{10\sin(2\theta)} d\theta \quad = 4\int_0^{\pi/2} \frac{(10\sin(2\theta))^3}{3}(\cos\theta+\sin\theta)\,d\theta$$

$$= \frac{4000}{3}\int_0^{\pi/2} \sin^3(2\theta)(\cos\theta+\sin\theta)\,d\theta \rightarrow \sin^3(2\theta) = (2\sin\theta\cos\theta)^3$$

$$= \frac{32000}{3}\int_0^{\pi/2} \sin^3\theta\cos^4\theta + \sin^4\theta\cos^3\theta\,d\theta$$

$$= \frac{32000}{3}\int_0^{\pi/2} \underbrace{\sin^2\theta}_{(1-\cos^2\theta)}\cos^4\theta\sin\theta\,d\theta + \frac{32000}{3}\int_0^{\pi/2} \sin^4\theta\underbrace{\cos^2\theta}_{(1-\sin^2\theta)}\cos\theta\,d\theta$$

$$= \frac{32000}{3}\int_0^{\pi/2}(\cos^4\theta-\cos^6\theta)\sin\theta\,d\theta + \frac{32000}{3}\int_0^{\pi/2}(\sin^4\theta-\sin^6\theta)\cos\theta\,d\theta$$

$$= \frac{32000}{3}\left[\left(\frac{-\cos^5\theta}{5}-\frac{\cos^7\theta}{7}\right)\Big|_0^{\pi/2} + \left(\frac{\sin^5\theta}{5}-\frac{\sin^7\theta}{7}\right)\Big|_0^{\pi/2}\right]$$

$$= \frac{32000}{3}\left[\left(-\frac{1}{7}+\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)\right]$$

$$\boxed{V = 1219.05\ ft^3}$$

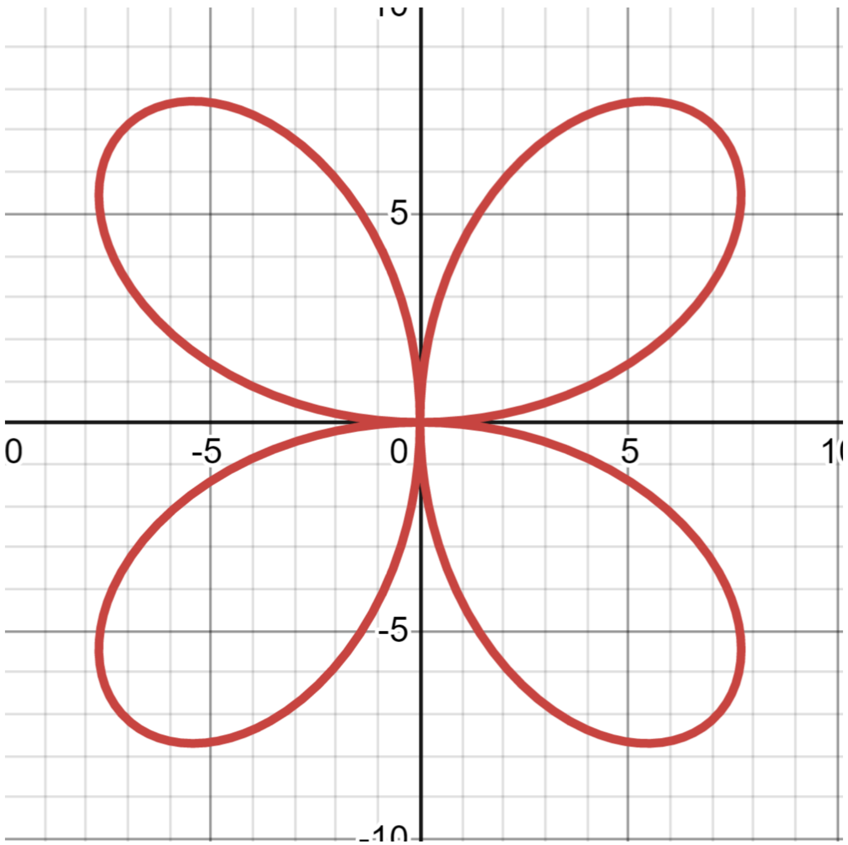
$$Average\ height = \frac{V}{A}$$

$$A = \int_{\theta 1}^{\theta 2} \int_{r1}^{r2} r dr d\theta$$

$$A = 4 \int_{0}^{\frac{\pi}{2}} \int_{0}^{10 sin 2\theta} r dr d\theta$$

$$A = 157.08\, ft^2$$

$$\frac{V}{A} = \frac{1219.05 ft^3}{157.08 ft^2} = \boxed{7.76 ft}$$



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$$A = 4\int_0^{\pi/2}\int_0^{10\sin(2\theta)} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \frac{100\sin^2(2\theta)}{2}\,d\theta = 200\int_0^{\pi/2} \sin^2(2\theta)\,d\theta$$

$$= 100\int_0^{\pi/2} 1 - \cos(4\theta)\,d\theta$$

$$= 100\left[\theta - \frac{1}{4}\sin(4\theta)\Big|_0^{\pi/2}\right] = 50\pi$$

$$\boxed{A = 157.08\ ft^2}$$

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Conclusion

- The calculus of clovers thus have many applications in fundamental sciences, engineering and transportation

- We show how multivariable calculus studies using polar and cylindrical coordinates help unravel the characteristics of these shapes.

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