Mathematical Modeling and Kinematics of a Glider

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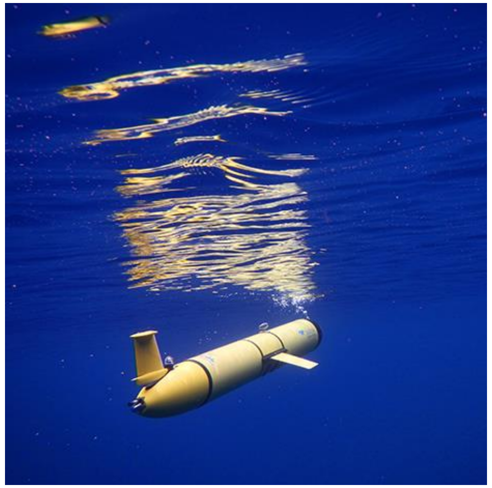
Introduction

- Here we will examine the mathematics behind glider's flight and their trajectories.
- We will look at how to identify characteristics of the flight, such as finding instantaneous velocity and acceleration through analytical and numerical methods.
- Lastly, we will show how to rotate and translate gliders using matrices.

Glider: Aircraft or vessel without an engine

Examples: hang gliders, underwater gliders



Glider Basics

1. Throwing the aircraft → Initial Velocity

2. Catapult made from rubber bands and tow lines → Velocity & Altitude

3. Pilots run and jump off the cliff → Velocity & Altitude

4. Towed aloft by powered aircraft and then cut loose.

Glider Basics

- Air gliders are often given an initial height by a powered aircraft that brings the glider in the air.
- Gliders can exchange the potential energy of an initial height for kinetic energy by flying to a lower altitude. This increases velocity.
- Gliders are always flying downward in comparison to the medium in which they are flying.

Minimizing Turbulence

- The flightpath of a hang glider is modeled using a polynomial function
- Its parameters are optimized using calculus-based techniques to minimize turbulence.

Flightpath of Hang Glider

Minimizing Turbulence

If the initial flightpath of a hang glider is:

$$f(x) = x^4 - \frac{28}{3}x^3 + 2ax^2$$

Where α depends on a certain physical properties of the hang glider and other environmental conditions. To avoid significant turbulence which may destabilize the glider, the flight path should not have local maxima.

How many positive integers (α <1000) are there such that it has no local maxima?

Minimizing Turbulence

$$f(x) = x^4 - \frac{28}{3}x^3 + 2ax^2$$

Maxima /Minima when $f'(x)$ is 0

$$f'(x) = 4x^3 - 28x^2 + 4ax = 0$$

$$= 4x(x^2 - 7x + a) = 0$$

$$x = \frac{7 \pm \sqrt{49 - 4a}}{2}$$

$$49 \geq 4a$$

$$a \leq 12.25$$

For α = 1, 2, …, 12, f'(x) has extreme turbulence

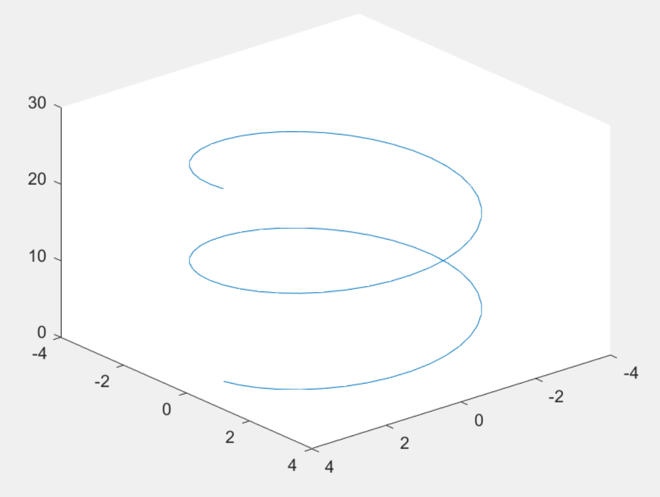
For α = 13, 14, …, 999, no real zeros

987 Values of α
(no turbulence)

Vector-Valued Functions

If a person on a hang glider followed the path of the helix below, we can find the instantaneous velocity and acceleration equations.

$$\vec{r}(t) =< 3\cos(t)\,, 3\sin(t)\,, 2t >$$



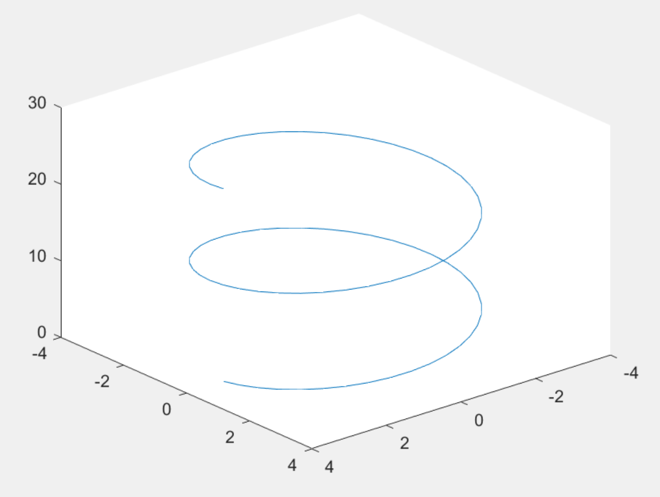
$$\vec{v}(t) = \vec{r}'(t) =< -3\sin(t)\,, 3\cos(t)\,, 2 >$$

$$\vec{a}(t) = \vec{r}''(t) =< -3\cos(t)\,, -3\sin(t)\,, 0 >$$

Vector-Valued Functions

We can also find when the glider's acceleration is orthogonal to its velocity.

$$\vec{v} \cdot \vec{a} = 0$$



$$< -3\sin(t), 3\cos(t), 2 > \cdot < -3\cos(t), -3\sin(t), 0 >= 0$$

$$9\cos(t)\sin(t) - 9\cos(t)\sin(t) = 0$$

This happens at all values of t.

Vector-Valued Functions

Lastly, we can find the osculating plane of the glider. At point $(0,3,\pi)$. $t=\pi/2$

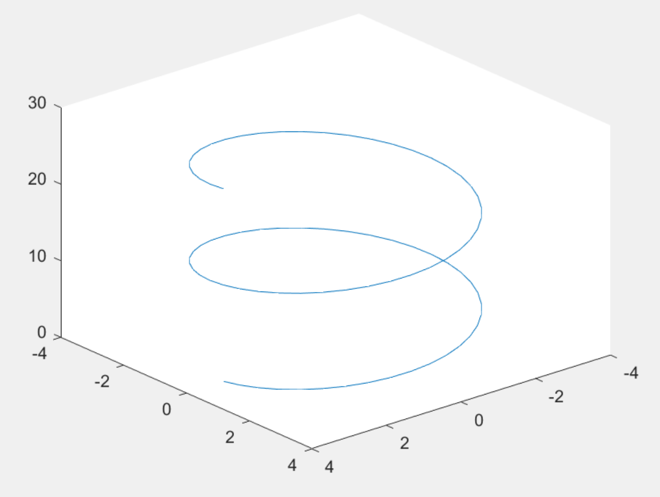
$$\vec{r}(t) = \langle 3\cos(t), 3\sin(t), 2t \rangle$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \left\langle \frac{-3\sin(t)}{\sqrt{13}}, \frac{3\cos(t)}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$$

$$\vec{N}(t) = \frac{r''(t)}{\|r''(t)\|} = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \left\langle \frac{2}{\sqrt{13}}\sin(t), -\frac{2}{\sqrt{13}}\cos(t), \frac{3}{\sqrt{13}} \right\rangle$$

$$\vec{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{2}{\sqrt{13}}, 0, \frac{3}{\sqrt{13}} \right\rangle$$



$$\frac{2}{\sqrt{13}}(x-0) + 0(y-3) + \frac{3}{\sqrt{13}}(z-\pi) = 0$$

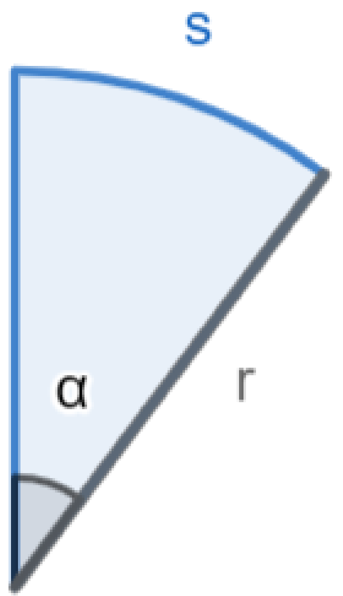
$$\boxed{\frac{2}{\sqrt{13}}x + \frac{3}{\sqrt{13}}z = \frac{3\pi}{\sqrt{13}}}$$

Modeling an Underwater Glider

- Underwater gliders, in contrast to air gliders, operate underwater collecting conductivity, temperature, and depth data.
- We modeled the path of an underwater glider using vector calculus to numerically find the instantaneous velocity and acceleration.

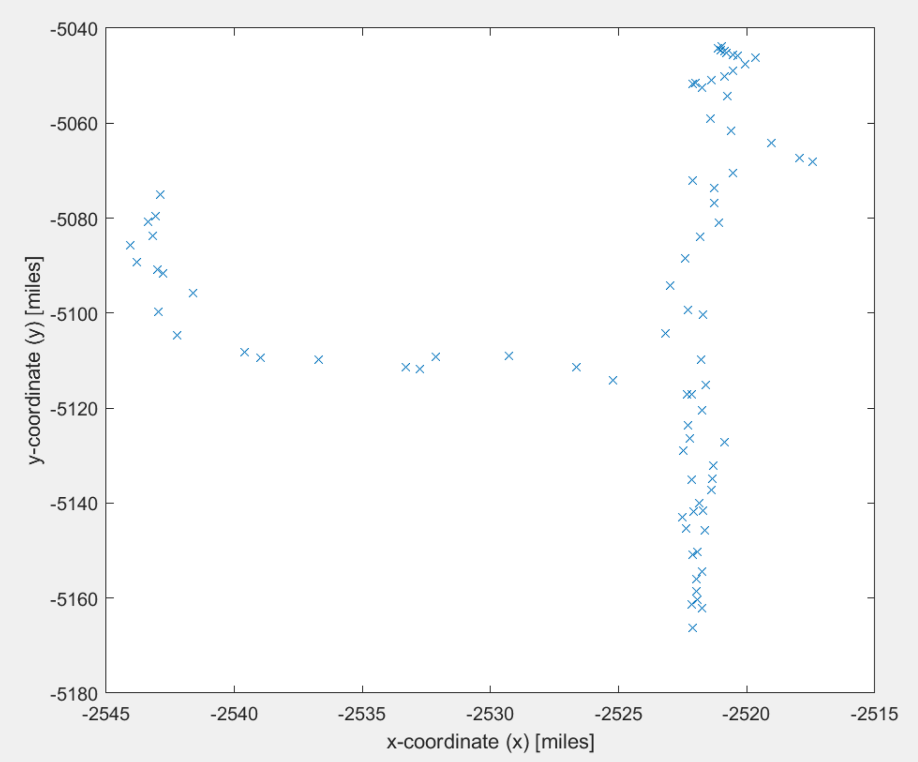
Modeling an Underwater Glider

- We used the latitude and longitude underwater glider position data from the University of Washington to map its trajectory.



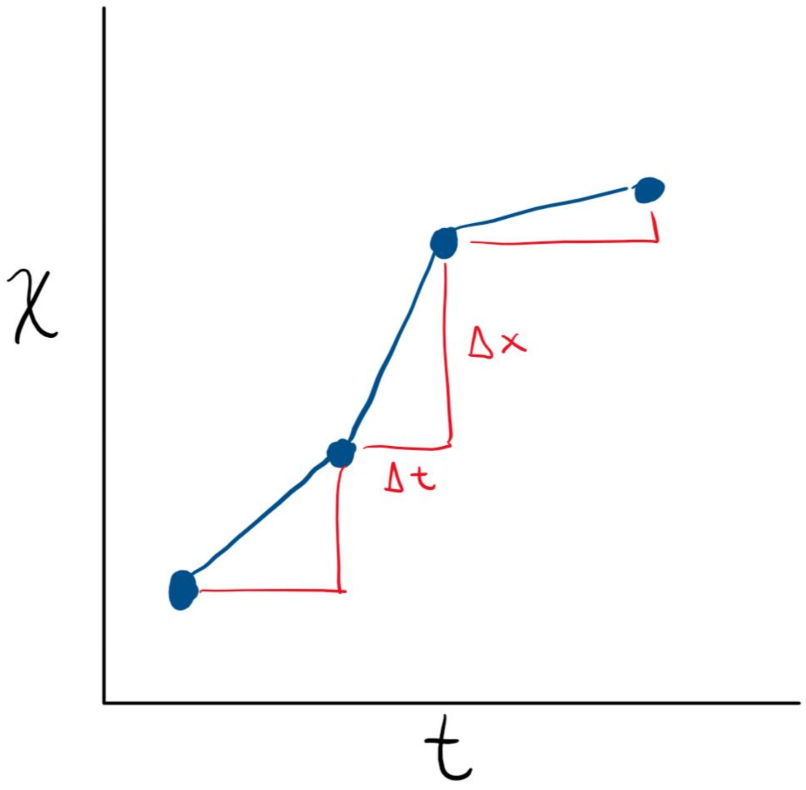
$$\frac{s}{r} = \alpha$$

$$s = \alpha \cdot r$$



Instantaneous Velocity and Acceleration

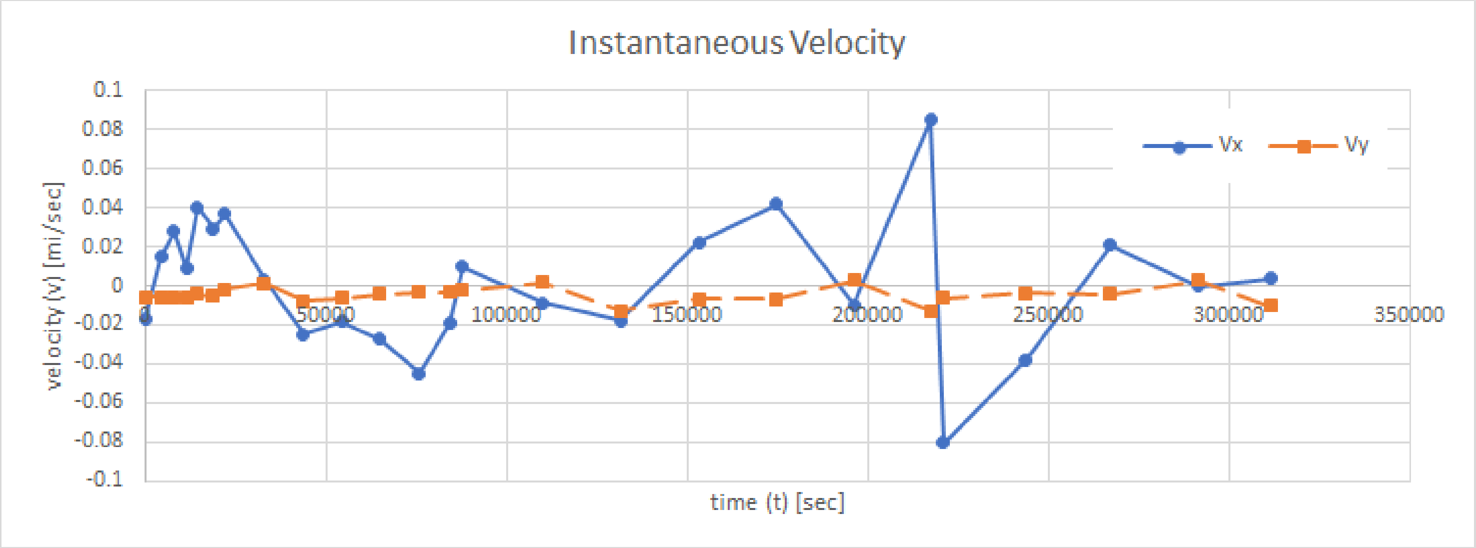
- By taking the average slope between adjacent data points, we can approximate the instantaneous velocity in the x and y coordinates.

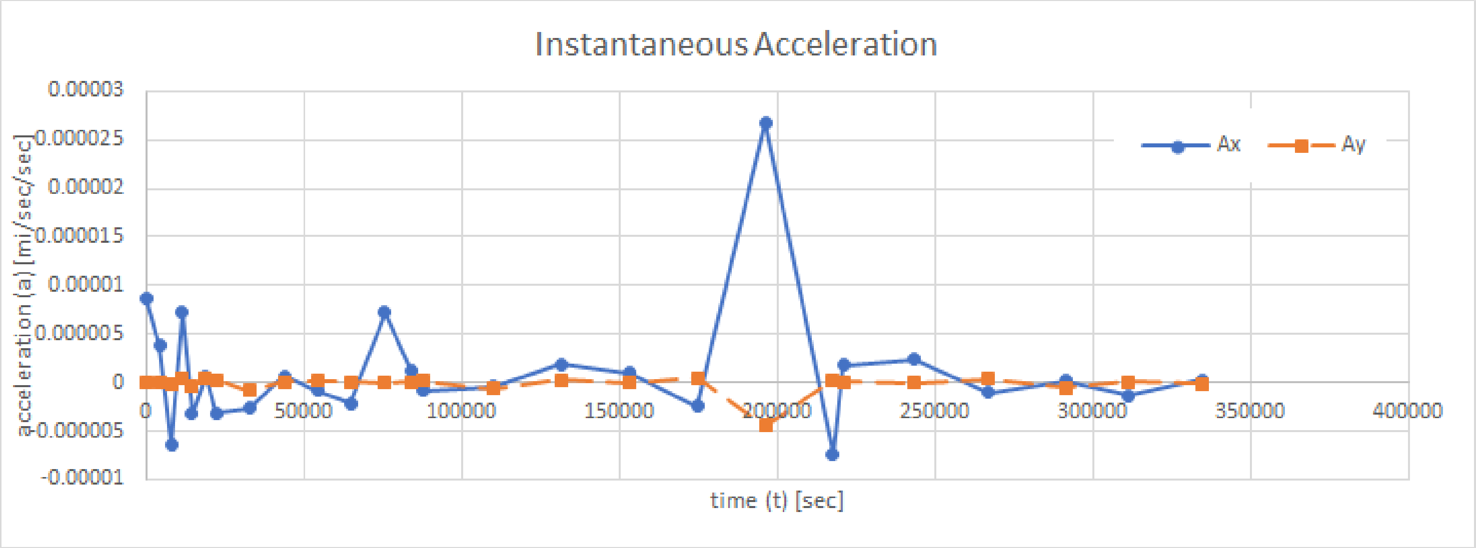


$$v_x = \frac{x_2 - x_1}{t_2 - t_1}$$

$$a_x = \frac{v_{x2} - v_{x1}}{t_2 - t_1}$$

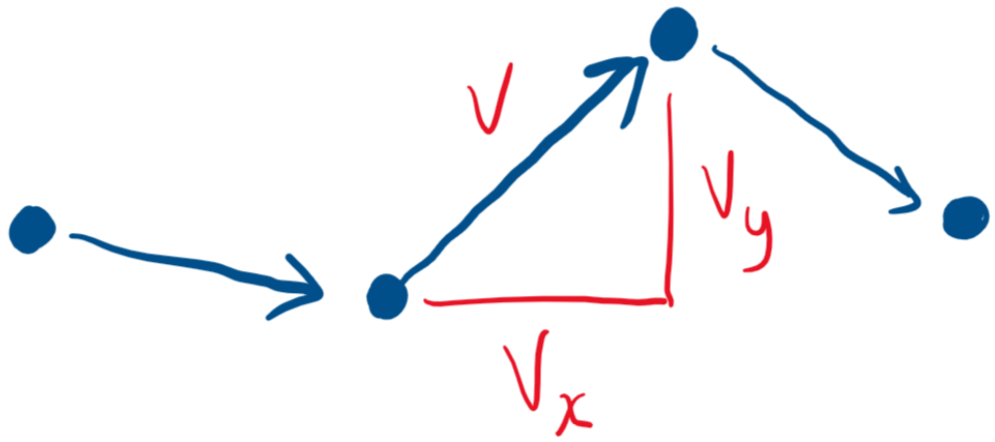
Instantaneous Velocity and Acceleration





Magnitude of Velocity and Acceleration

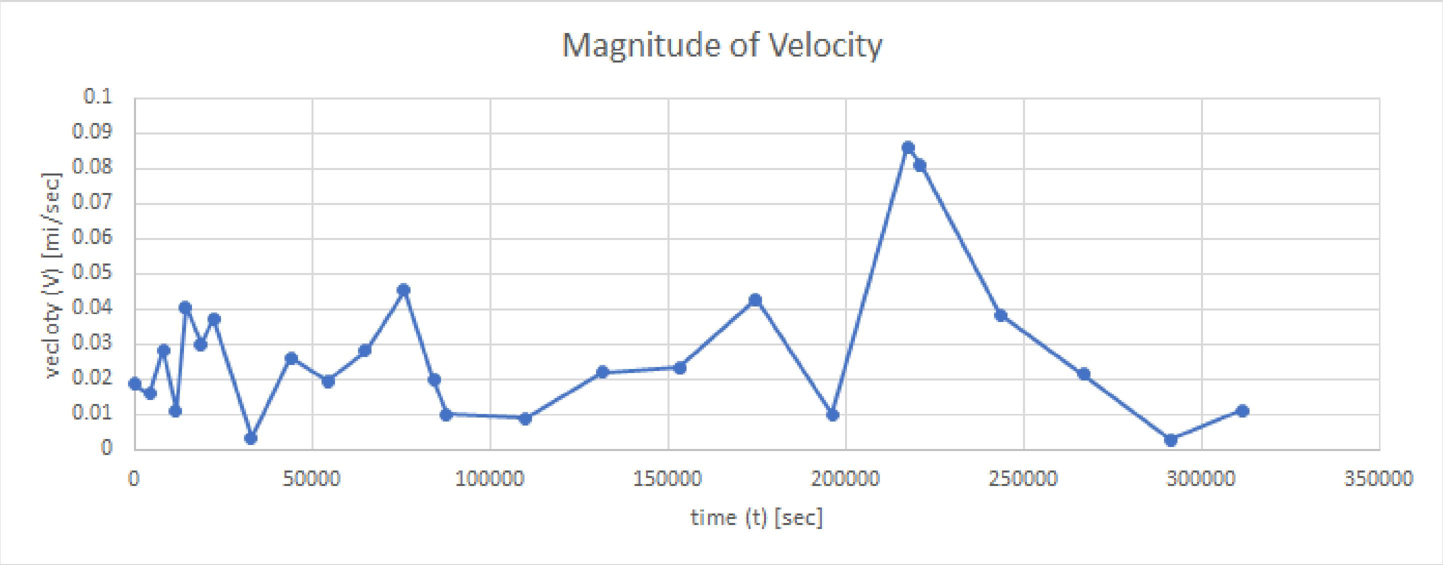
- We can then find the magnitude of the velocity and acceleration vectors.

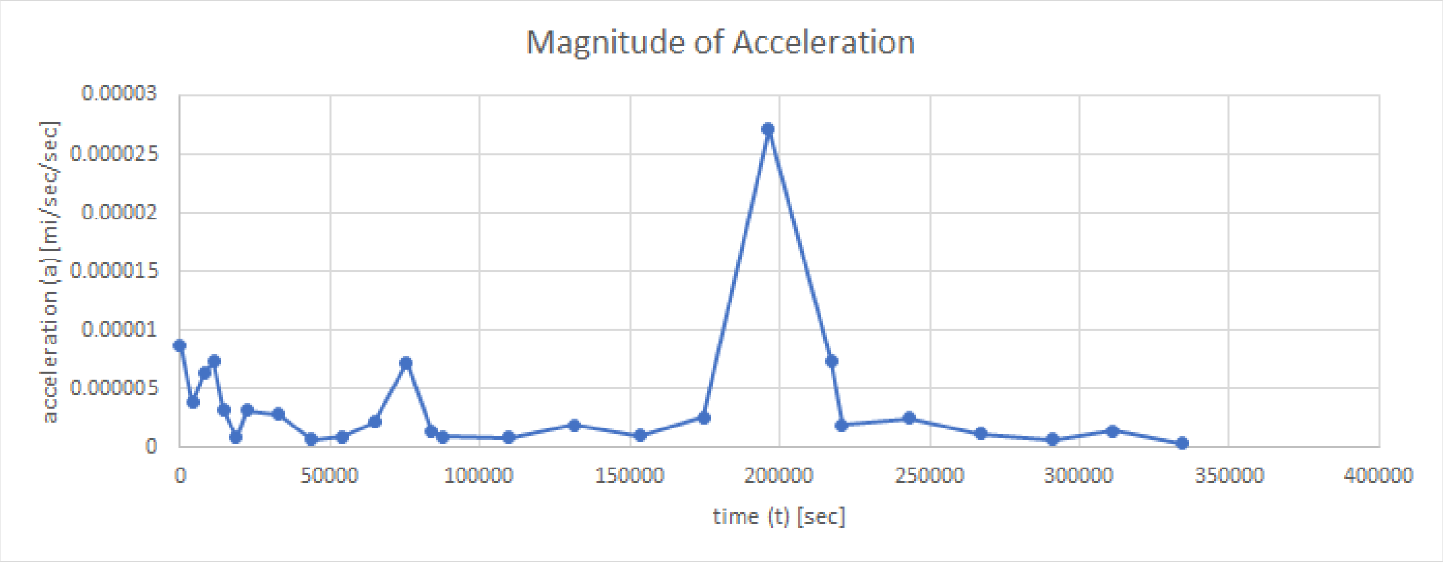


$$V = \sqrt{v_x^2 + v_y^2}$$

$$A = \sqrt{a_x^2 + a_y^2}$$

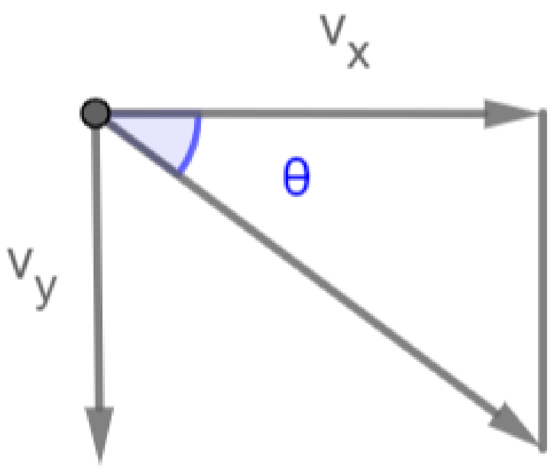
Magnitude of Velocity and Acceleration





Finding the Orientation of the Glider

- We can use this to also find the orientation of the glider at a certain time.



$$\theta = \arctan\left(\frac{V_y}{V_x}\right)$$

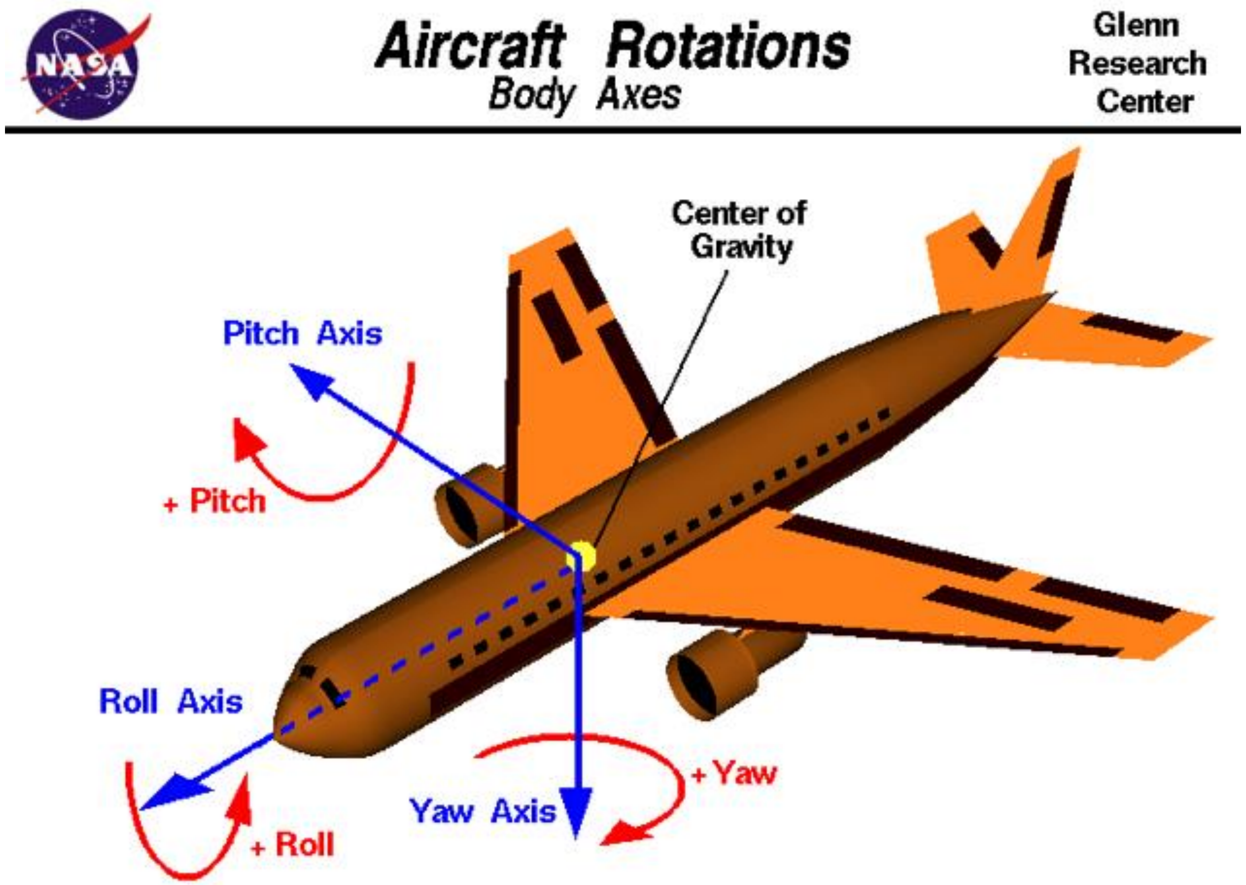
At t=8013 seconds:

$$\theta = \arctan\left(\frac{-0.1955\ ft}{0.0479\ ft}\right) = -76.233°$$

76.233° south of east

Translating and Rotating Gliders

- For a glider to be effective, one needs to tweak its degrees of freedom to collect data from various depths or points
- Done using translations and rotations
- We use linear algebra to derive the transformation matrices for simple translations and rotations in three dimensions



Translations

The translation about x, y, and z axes of l is then in the form:

$$T_x = \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix}, \quad T_y = \begin{bmatrix} 0 \\ l \\ 0 \end{bmatrix}, \quad T_z = \begin{bmatrix} 0 \\ 0 \\ l \end{bmatrix}.$$

Rotation Matrices

$$R_z(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Rotation by angle θ about the z-axis

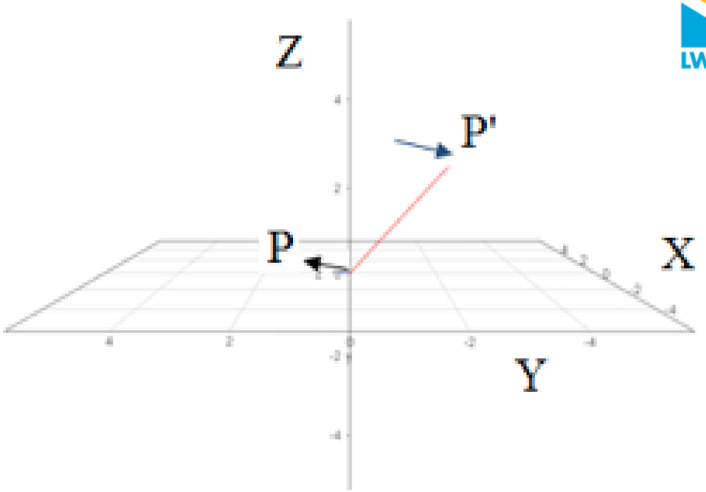
$$R_y(\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix},$$

Rotation by angle θ about the y-axis

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Rotation by angle θ about the x-axis

Coordinate Transformation



Consider the point P rotated by 30° about the x-axis and translated by (2,-1,3).

Rotation matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Transformation: (Rotation only)

$$(x, y, z) \longrightarrow \left(x, \frac{\sqrt{3}\, y}{2} - \frac{z}{2}, \frac{y}{2} + \frac{\sqrt{3}\, z}{2}\right)$$

Transformed point P’ (rotation + translation):

$$\left(3, -1 - \frac{\sqrt{3}}{2}, \frac{5}{2}\right)$$

The gliders data is collected by changing its orientation in the water.

The glider's pitch, yaw, and roll can be changed remotely by an operator

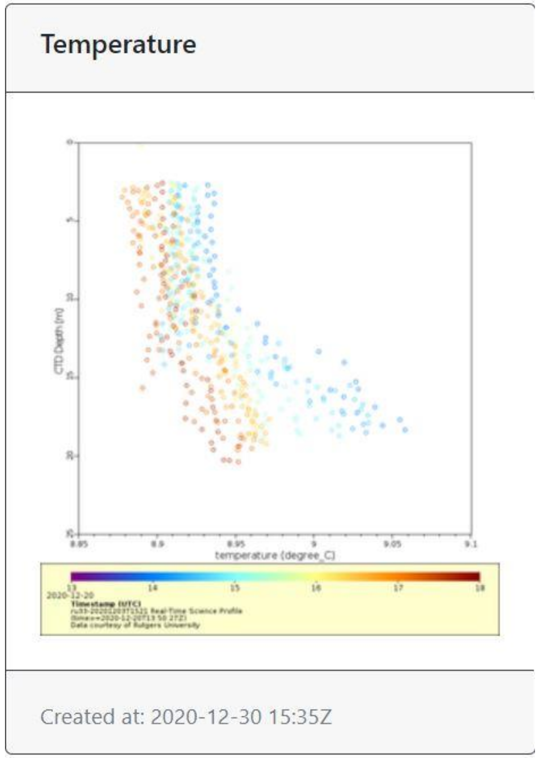
Collected data is an integral part to understanding the ocean in real-time

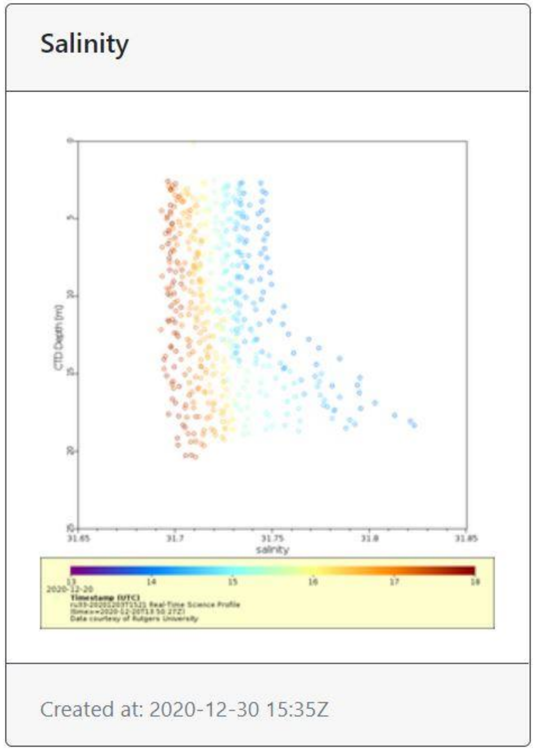
The glider communicates via an Iridium satellite link every time the glider surfaces, which can occur several times per day.

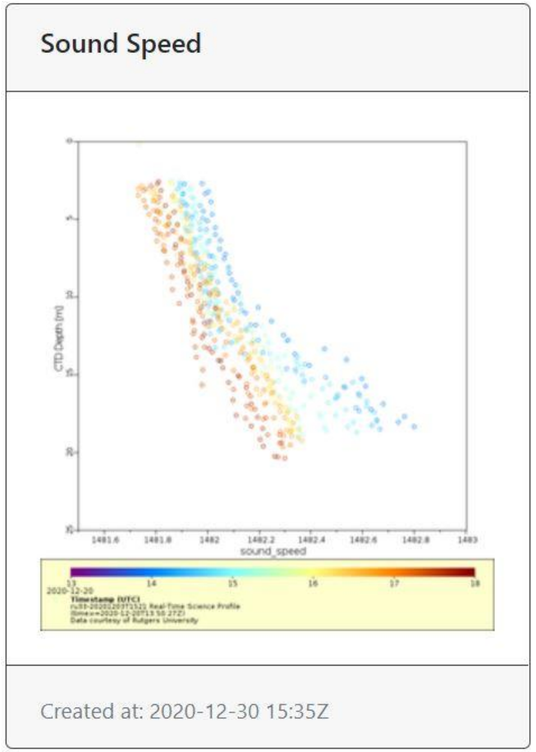
https://rucool.marine.rutgers.edu/data/underwater-gliders/

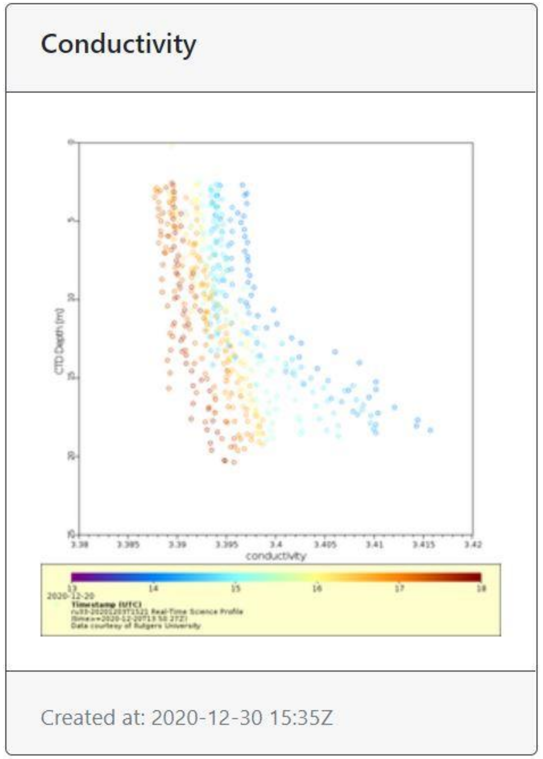
Real Time Data

- Data received by the gliders are translated into Real time data graphs. Received several times per day as the glider surfaces
- The data received assists in accurately modeling oceanic forecasts and ground truthing of ocean color satellite algorithms
- The COOL group aims to increase the number of gliders in the ocean allowing for a more continues stream of data









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Conclusion

- We used calculus-based optimization techniques, matrix algebra, and numerical differentiation to analyze and interpret real data for gliders.

- This research project provides a novel avenue for hands-on oceanic exploration, and application of key mathematical concepts and data visualization.

References

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NASA online educational resources on gliders

https://www.grc.nasa.gov/www/k-12/airplane/glider.html

Flightpath of a glider

https://www.grc.nasa.gov/www/k-12/airplane/glidang.html

UW APL data online

http://www.bco-dmo.org/dataset-deployment/485603

Underwater glider modelling and analysis for net buoyancy, depth and pitch angle control, Nur Afande Ali Hussain, Mohd Rizal Arshad, Rosmiwati Mohd-Mokhtar, Ocean Engineering 38 (2011) 1782–1791

Video Presentation of Project: https://www.youtube.com/watch?v=9vEF9a80IPM