Velocity and Acceleration Profiles of Space Shuttles

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Introduction

- In this project, we fitted a 4th order polynomial to the STS-121 shuttle data to model its ascent.

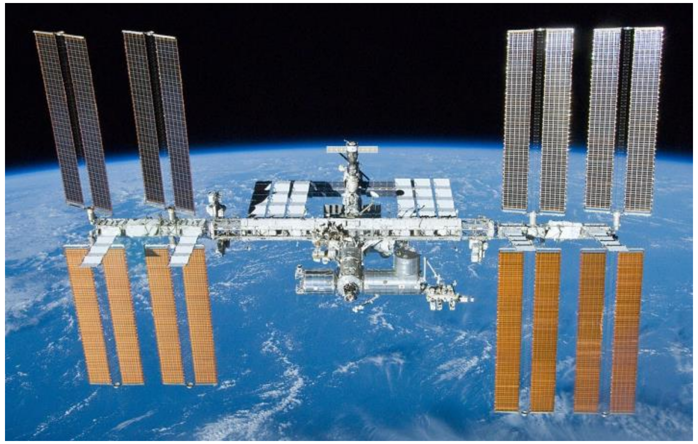
- We used this model to find the local extrema using Newton's Method, and the inflection points using the second derivative of the function.

- Additionally, we optimized solar panel positioning and geometry on a satellite.

Introduction

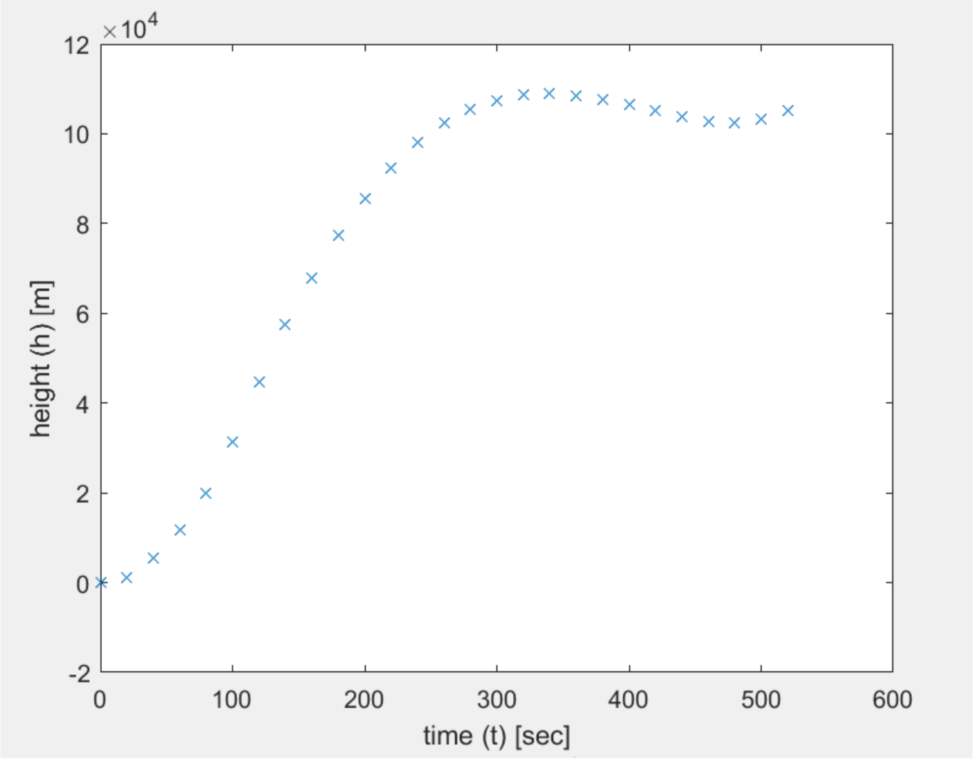
- Space shuttles use two solid rocket boosters and the Shuttles main engines to produce the required force to achieve low earth orbit.

- The STS-121 mission was headed to the International Space Station.



Data Fitting

- The ascent data for the shuttle launch is given by NASA.



t	h(t)
0	-8
20	1244
40	5377
60	11617
80	19872
100	31412
120	44726
140	57396
160	67893
180	77485
200	85662
220	92481
240	98004
260	102301
280	105321
300	107449
320	108619
340	108942
360	108543
380	107690
400	106539
420	105142
440	103775
460	102807
480	102552
500	103297
520	105069

Data Fitting

- We can use the ascent data to fit a 4th order polynomial to model the launch.

$$h(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4$$

$$\begin{cases} h_0 = c_0 + c_1 t_0 + c_2 t_0^2 + c_3 t_0^3 + c_4 t_0^4 \\ h_1 = c_0 + c_1 t_1 + c_2 t_1^2 + c_3 t_1^3 + c_4 t_1^4 \\ \vdots \\ h_{27} = c_0 + c_1 t_{27} + c_2 t_{27}^2 + c_3 t_{27}^3 + c_4 t_{27}^4 \end{cases}$$

t	h(t)
0	-8
20	1244
40	5377
60	11617
80	19872
100	31412
120	44726
140	57396
160	67893
180	77485
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500	103297
520	105069

Data Fitting

$$\vec{c} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 20 & 400 & 8000 & 160000 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 520 & 270400 & 14060800 & 73116160000 \end{bmatrix}$$

$$h = \begin{bmatrix} -8 \\ 1244 \\ \vdots \\ 105069 \end{bmatrix}$$

Least square minimizing error method (approximate solution)

$$A\vec{c} = h$$

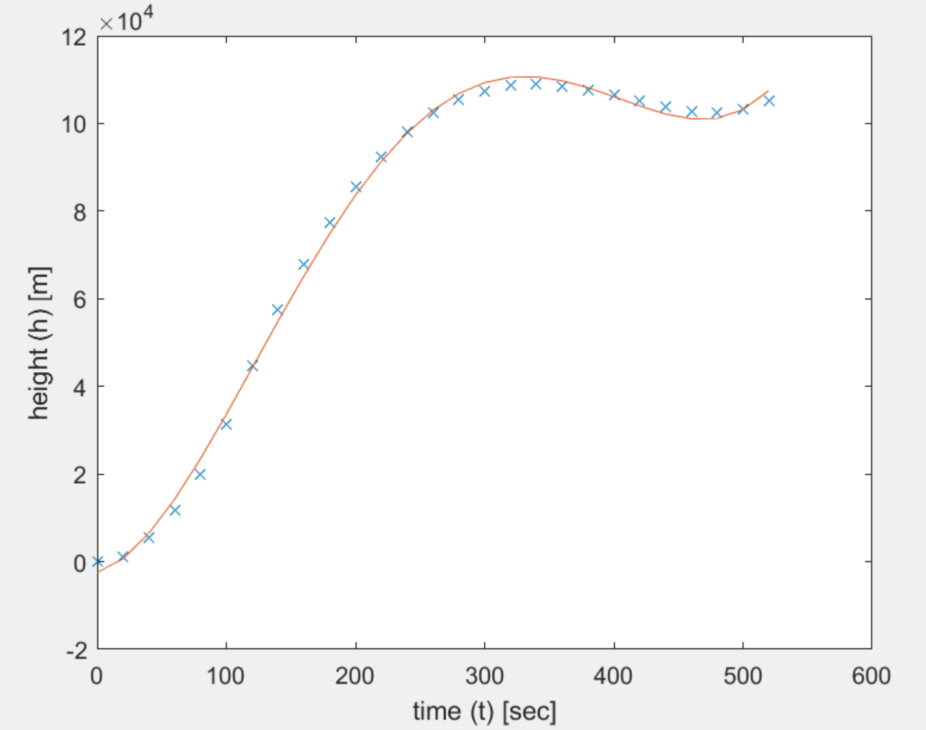
$$A^T A\vec{c} = A^T h$$

$$\vec{c} = (A^T A)^{-1} A^T h$$

Data Fitting

$$\vec{c} = (A^T A)^{-1} A^T h$$

$$\vec{c} = \begin{bmatrix} -2475.79932432985 \\ 79.7543864362582 \\ 4.15083689560658 \\ -0.01479121727846 \\ 0.0000140311060867044 \end{bmatrix}$$



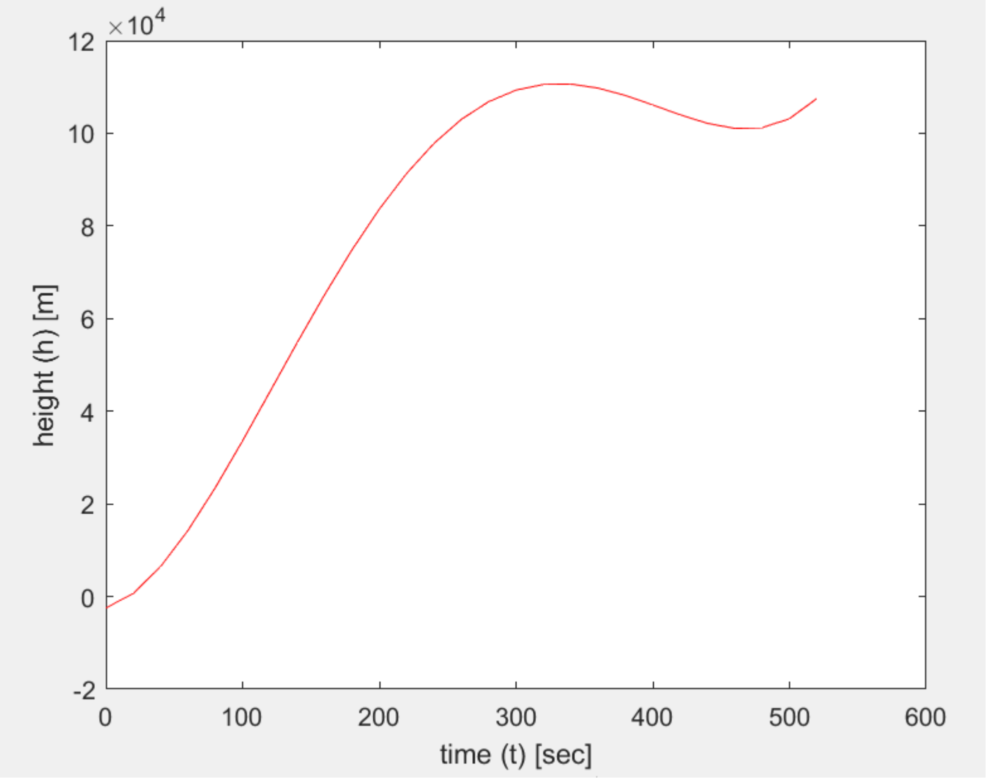
$$h(t) = -2475.799324 + 79.75438644t + 4.150836896t^2 - 0.014791217t^3 + 0.0000140311t^4$$

$$r^2 = 0.9978$$

Finding Local Extrema

$h(t) = -2475.799324 + 79.75438644t + 4.150836896t^2 - 0.014791217t^3 + 0.0000140311t^4$

Height



$h'(t) = 0$ at local extrema

$$h'(t) = 79.7544 + 8.30167t - 0.044374t^2 + 0.000056t^3$$

$$79.7544 + 8.30167t - 0.044374t^2 + 0.000056t^3 = 0$$

Finding Local Extrema

$$79.7544 + 8.30167t - 0.044374t^2 + 0.000056t^3 = 0$$

Newton's Method to find zeros

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$$

$$t_{n+1} = t_n - \frac{79.7544 + 8.30167t_n - 0.044374t_n^2 + 0.000056t_n^3}{8.30167 - 0.088747t_n + 0.000168t_n^2}$$

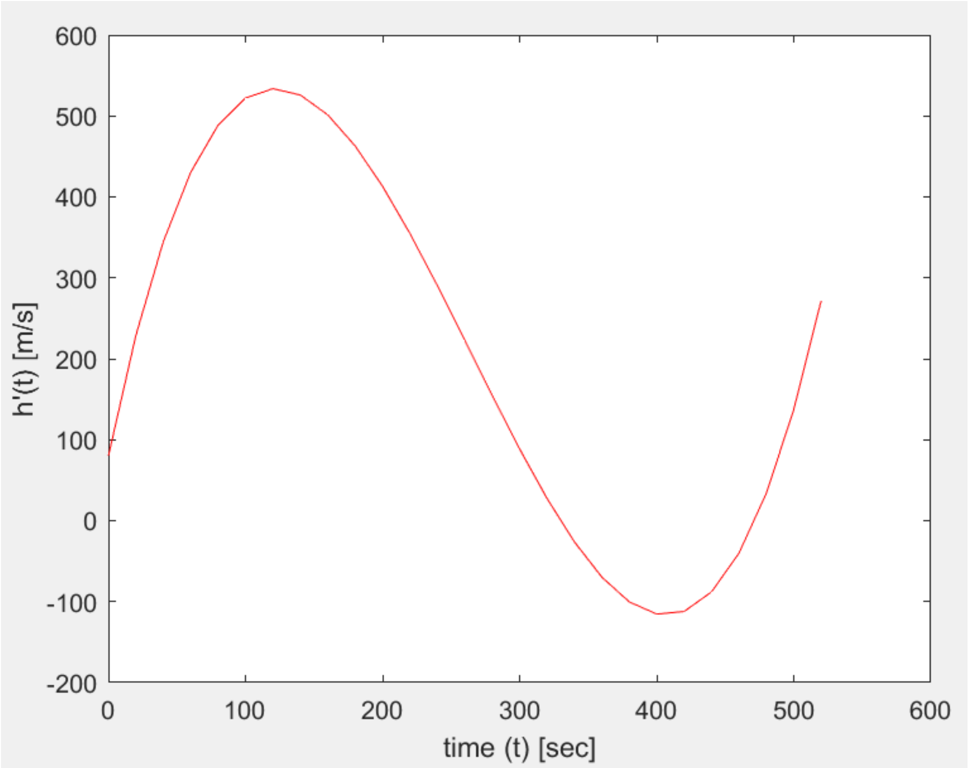
$t_0 = 300$

$t = 331.50398872648$

$t_0 = 500$

$t = 468.28025184524$

h'(t)



Finding Local Extrema

$$t = 331.50398872648 \qquad t = 468.28025184524$$

Use concavity to determine point type

If h''(t) < 0 then concave down

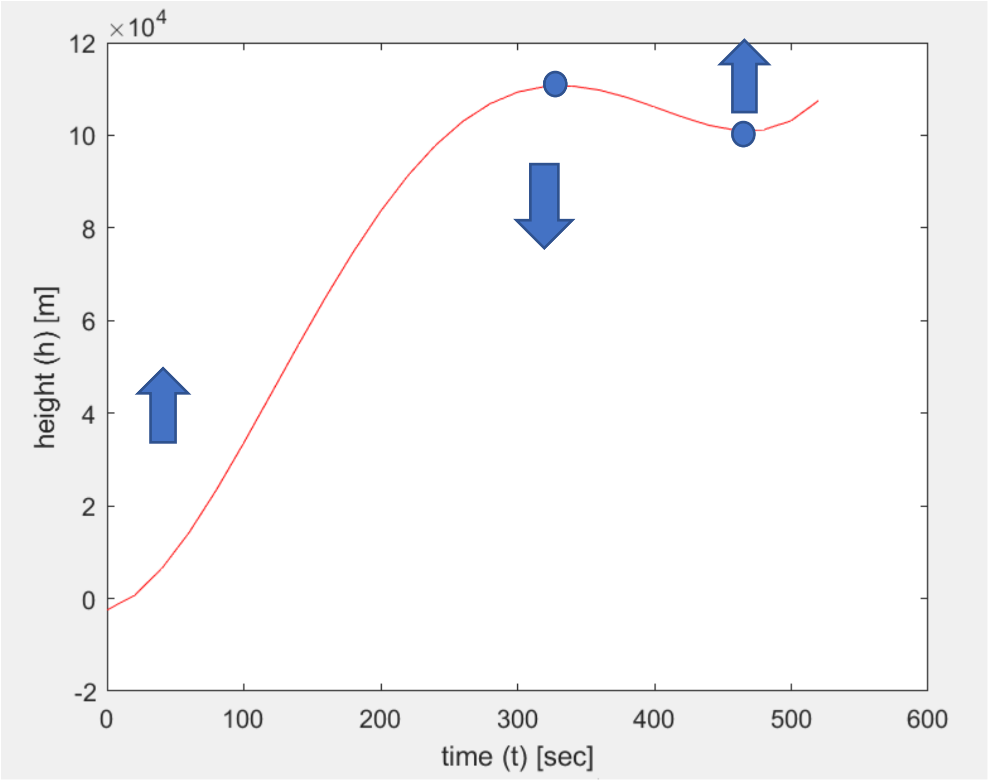
If h''(t) > 0 then concave up

$$h''(t) = 8.30167 - 0.088747t + 0.000168t^2$$

$h''(331.5) = -2.6$ concave down, local maximum

$h''(468.3) = 3.7$ concave up, local minimum

Height



Finding Inflection Points

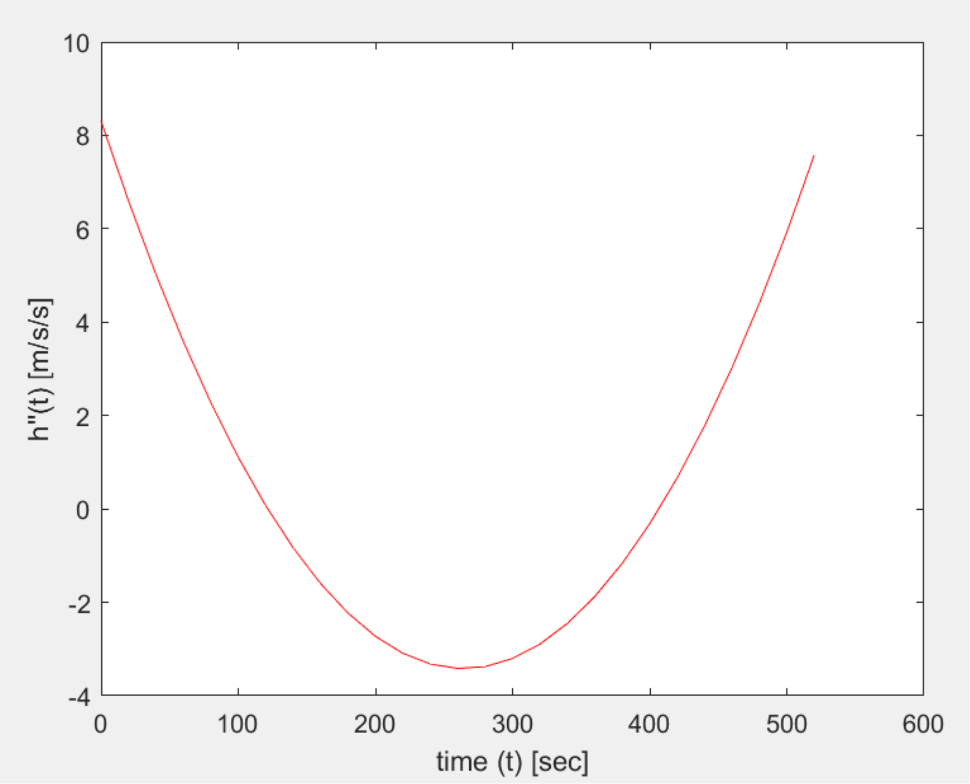
$$h(t) = -2475.80 + 79.75t + 4.15t^2 - 0.015t^3 + 0.00001t^4$$

$h''(t) = 0$ at inflection points

$$h''(t) = 8.30167 - 0.088747t + 0.000168t^2$$

$$8.30167 - 0.088747t + 0.000168t^2 = 0$$

h''(t)



Finding Inflection Points

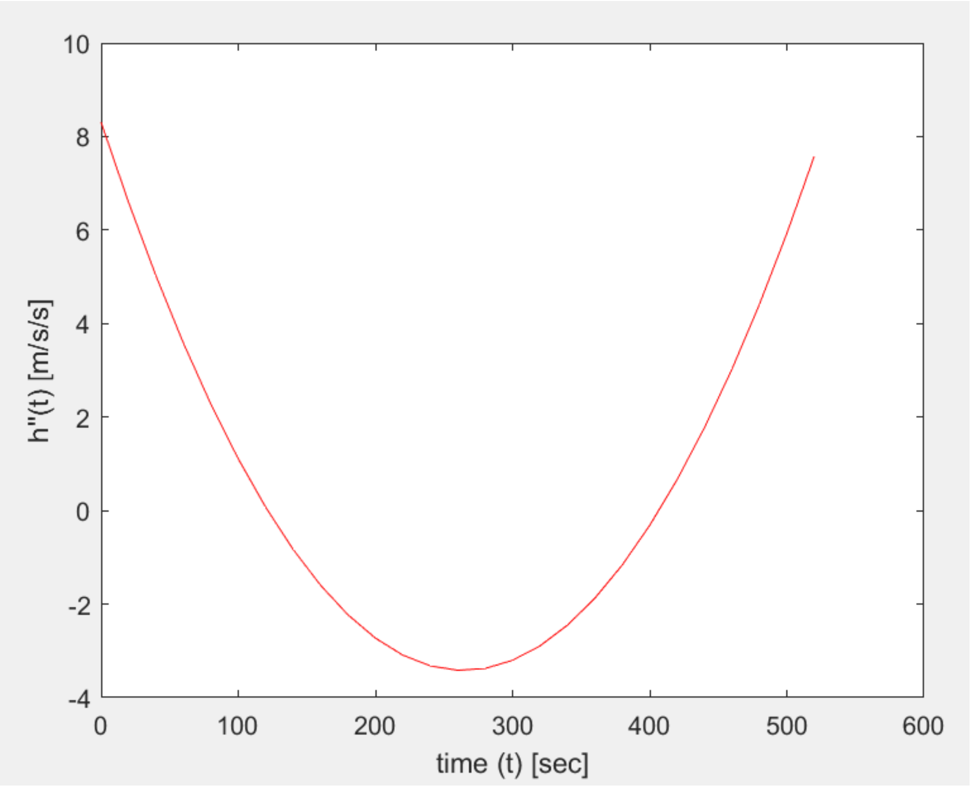
$$8.30167 - 0.088747t + 0.000168t^2 = 0$$

Quadratic equation to find zeros

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = 121.48, 406.78$$

h''(t)

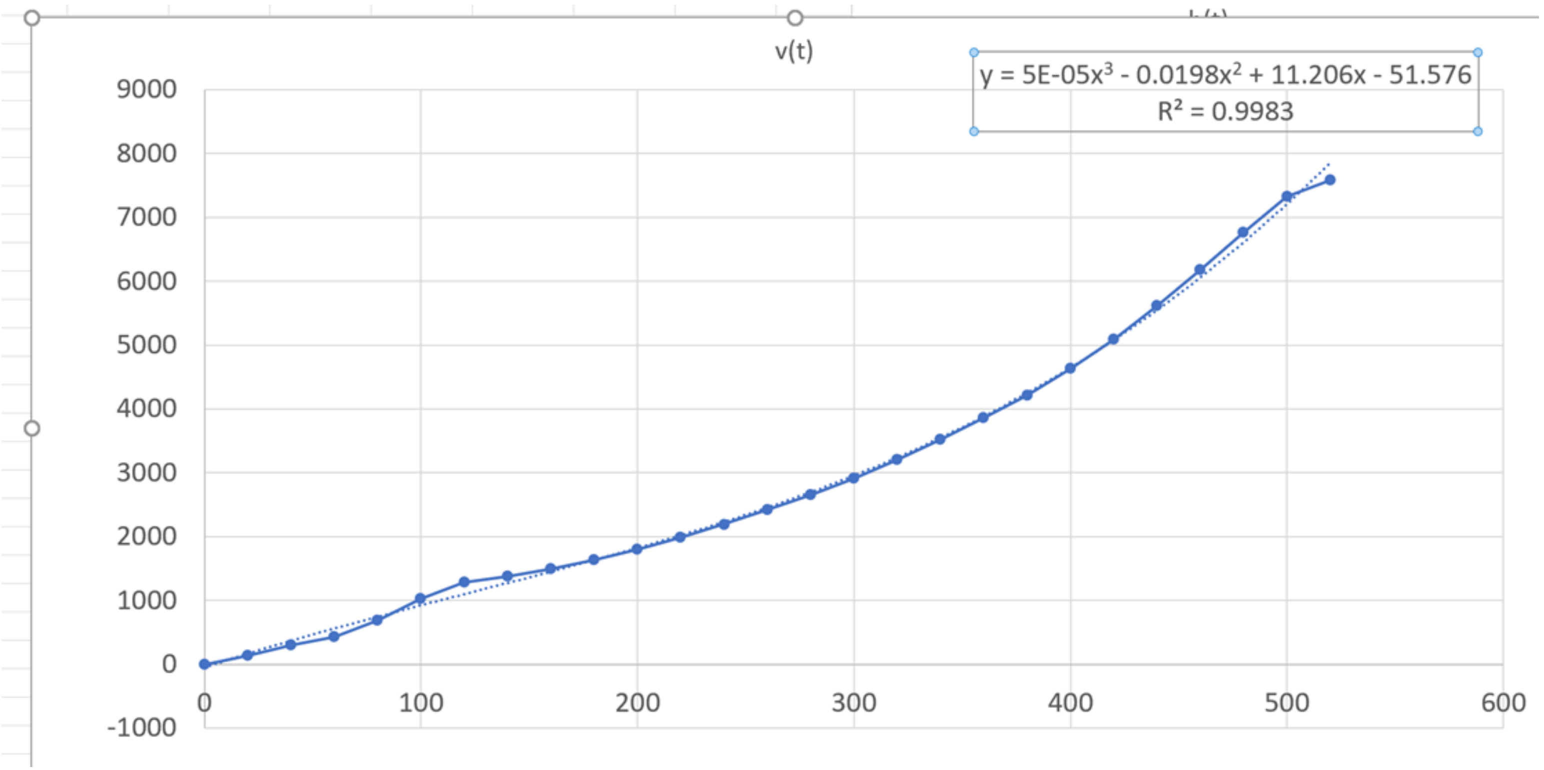


Average Velocity

We used the Integral Mean value theorem to calculate the shuttle's average velocity

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

$$Vave = \frac{1}{520}\int_0^{520} v(t)dt$$



Average Velocity

$$V_{ave} = \frac{1}{520}\int_0^{520} 5 \cdot 10^{-5}t^3 - 0.0198t^2 + 11.206t - 51.576\, dt$$

$$V_{ave} = 2835\,\frac{m}{s}$$

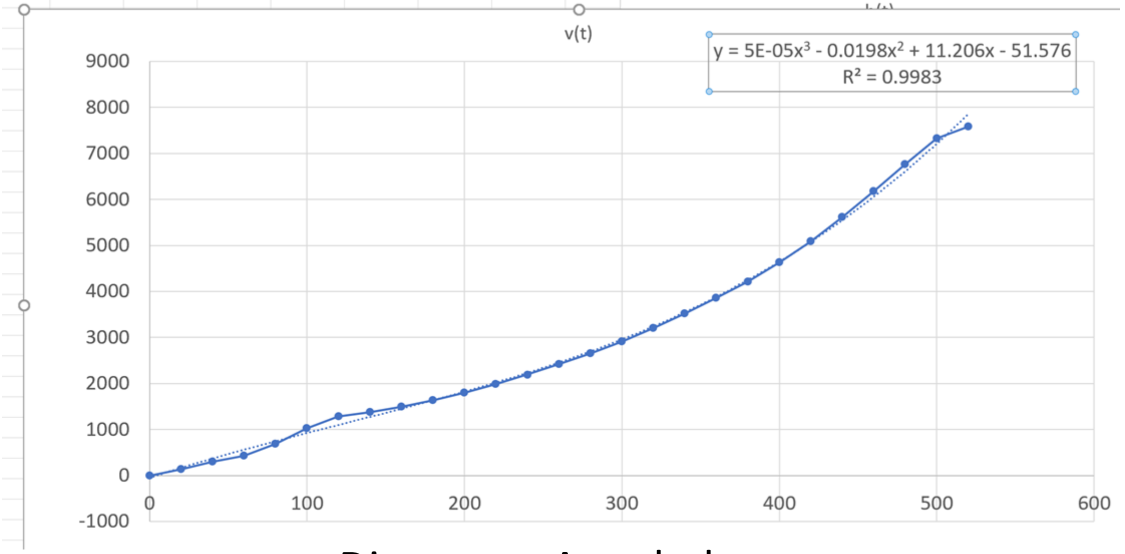
Distance travelled

$$Distance = \int_{t_1}^{t_2} |v(t)|\, dt$$

$$Distance = \int_0^{520} |5 \cdot 10^{-5}t^3 - 0.0198t^2 + 11.206t - 51.576|\, dt$$

$$= 1.47 \cdot 10^6\, m$$

$$= 1474\, km$$



Distance = Area below curve

Mass

- Table 1 shows the total mass of Discovery for mission STS-121 every 10 seconds from liftoff to SRB separation.
- Total mass includes the **orbiter, SRBs, ET, propellant, and payload**.
- You can see in the table that the space shuttle has a total mass of 2,051,113 kg at **t = 0**.
- After **2** minutes its total mass is only 880,377 kg, or **43 % of the original mass.**
- The burning of this vast amount of propellant is needed to get the space shuttle through Earth's atmosphere and into orbit.

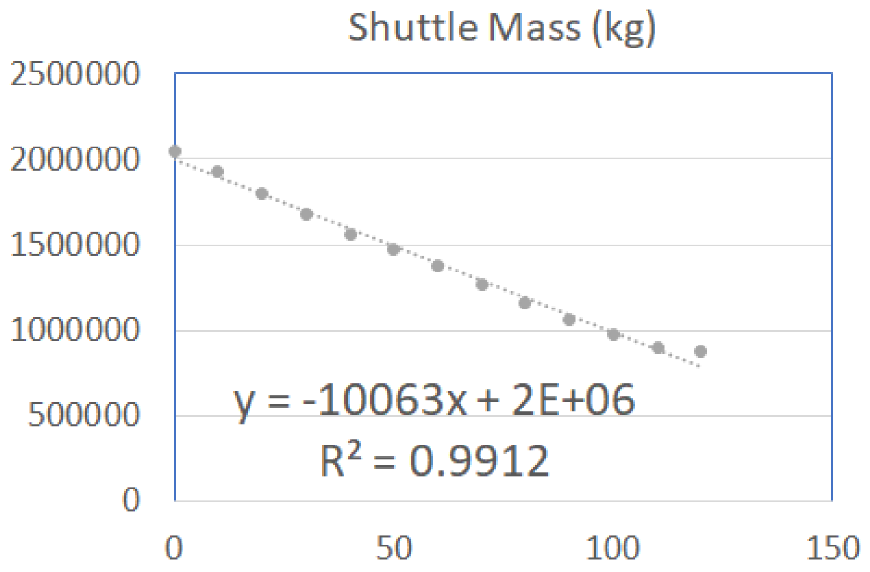
Table 1: STS-121 Discovery Ascent data (total mass)

Time (s)	**Space Shuttle Total Mass** (kg)
0	2,051,113
10	1,935,155
20	1,799,290
30	1,681,120
40	1,567,611
50	1,475,282
60	1,376,301
70	1,277,921
80	1,177,704
90	1,075,683
100	991,872
110	913,254
120	880,377

Mass

The plot on the right shows shuttle mass as a function of time.

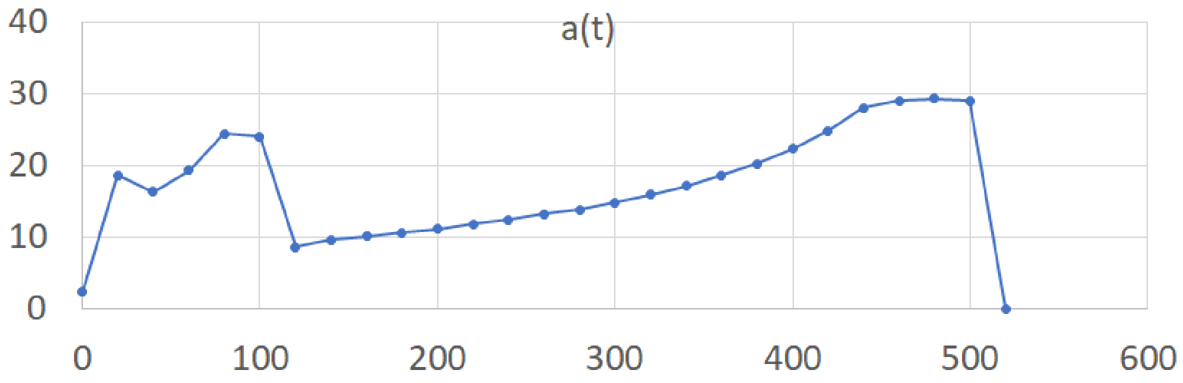
From 0-120 Sec, the mass is decreasing due to the burning of the propellent fuel in the booster



R= -0.9959
Negative correlation of mass (negative slope) as mass is decreasing with time, remaining **43% in 2 minutes**

Estimated percentage error=
|(Mass(calculated)-Mass(Observed))/Mass(Observed)|*100 = **2.42%**

Acceleration

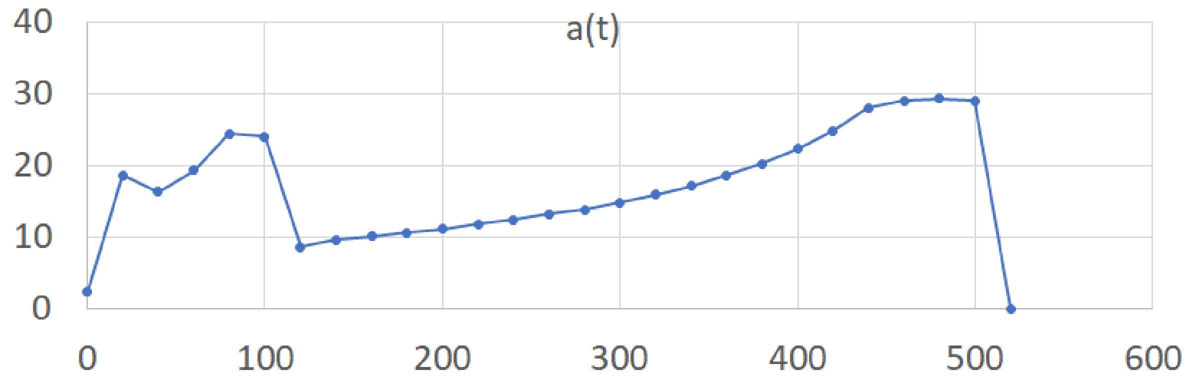


Between **t=0-120** sec, the Shuttle loses mass due to burning the propellent in the booster.

From **t=0 to 120** seconds, the space shuttle is burning the propellant in the Solid Rocket Boosters. This time interval is also where the space shuttle is in the denser (or thicker) part of the atmosphere.

During this time, the increasing dynamic pressure (or Q-Bar or simply Q) requires that the engines throttle down to about 70% to prevent damage to the space shuttle. Once Max Q is reached, the space shuttle throttles back to 100% and the acceleration increases again.

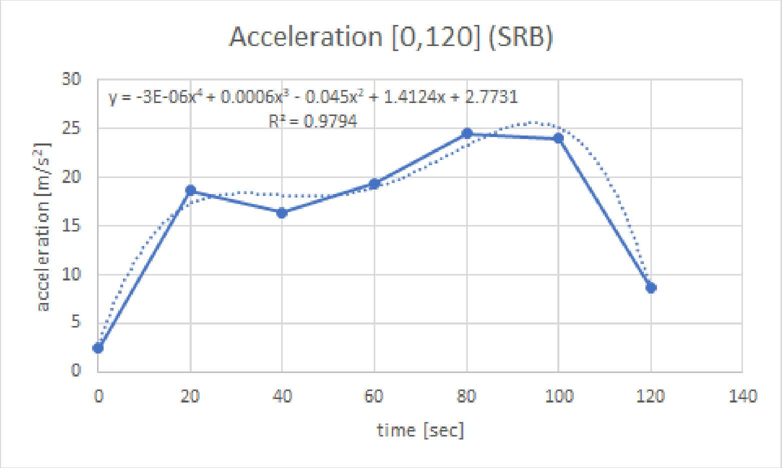
Acceleration



At 120 seconds the space shuttle has burned all the propellant from the SRB and they separate from the space shuttle. By burning more propellant in the external tank (reducing mass), the acceleration increases; and as the mass continues to decrease the acceleration increases at a faster rate until the space shuttle reaches its maximum acceleration of **3 g (29.4 m/s2)** at 450 seconds.

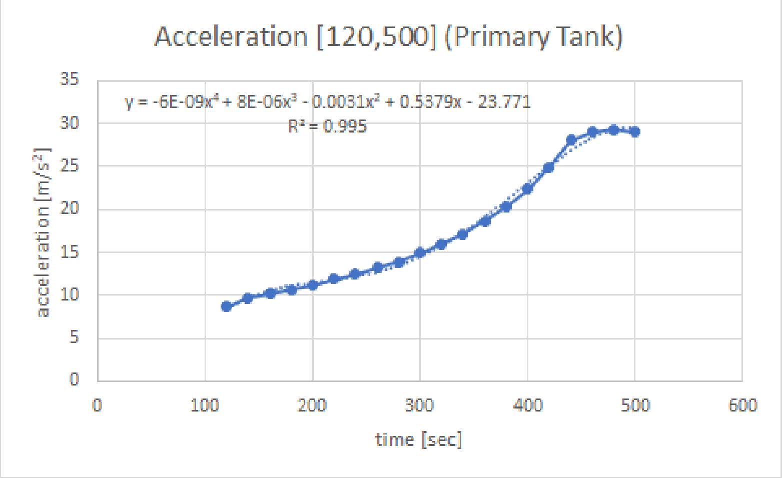
On the interval **[460,500]** the space shuttle is at max acceleration where it stays until it is ready for orbit.

Piecewise function



Stage 1, 2: 100% throttle – 70%, Propellent burned via Solid Rocket Booster (SRB)

$$\text{plot} \begin{cases} 1.236858\,x & 20 > x > 0 \\ -8.8889\,x & 40 > x > 20 \\ 6.60066\,x & 60 > x > 40 \\ 3.9216\,x & 80 > x > 60 \\ -40.8163\,x & 100 > x > 80 \\ -1.30804\,x & 120 > x > 100 \end{cases}$$

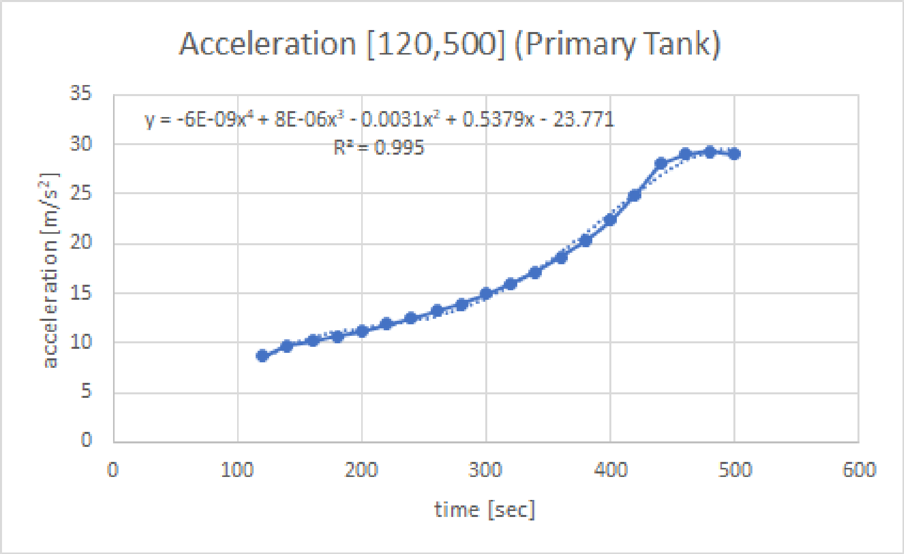


Stage 3: Propellent expended only via primary tank

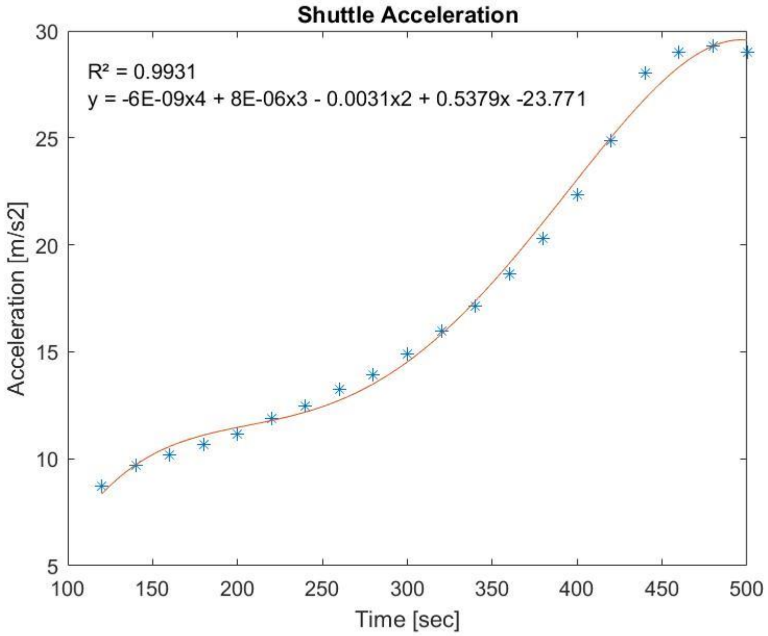
$$\text{plot} \begin{cases} 18.72844\,x & 500 > x > 120 \end{cases}$$

Regression: 4th Order Polynomial

Excel: $R^2 = 0.995$



MATLAB: $R^2 = 0.993$



MATLAB allows for us to create a fit with any order polynomial

Satellite Solar Panels

NASA Space Math
http://spacemath.gsfc.nasa.gov

A satellite is designed to fit inside the nose-cone (shroud) of a Delta II rocket. There is only enough room for a single satellite, so it cannot have deployable solar panels to generate electricity using solar cells. Instead, the solar cells have to be mounted on the exterior surface on the satellite. The satellite configuration is that of an octagonal prism. The total volume of the satellite is 10 cubic meters. The solar cells will be mounted on the octagonal top, bottom, and the rectangular side panels of the satellite.

Problem 1 - If the width of a panel is W, and the height of the satellite is H, what are the dimensions of the satellite that minimize the surface area and hence the available power that can be generated by the solar cells?

Problem 2 - If only 1/2 of the solar cells receive light at any one time, and the power they can deliver is 100 watts per square meter, what is the power that this satellite can provide to the experiments and operating systems?

Problem 1 - If the width of a panel is W, and the height of the satellite is H, what are the dimensions of the satellite that minimize the surface area and hence the available power that can be generated by the solar cells?

Volume of an octogonal prism:

$$V = 2(1+\sqrt{2})W^2H$$

$$10 = 2(1+\sqrt{2})W^2H$$

$$H = \frac{5}{(1+\sqrt{2})W^2}$$

Surface area of an octogonal prism:

$$S = 4(1+\sqrt{2})W^2 + 8WH$$

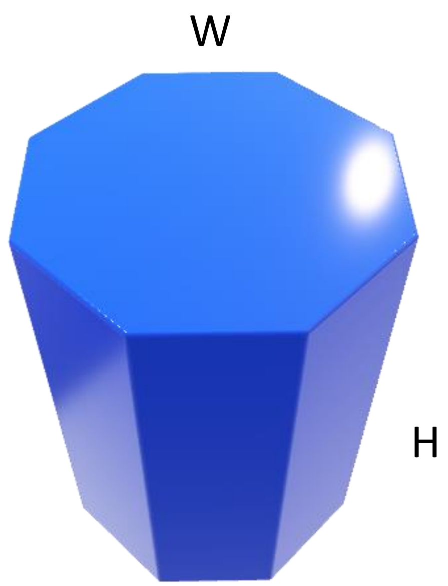
$$= 4(1+\sqrt{2})W^2 + \frac{40}{(1+\sqrt{2})W}$$

Minimum surface area for $V = 10$:

$$S'(W) = 8(1+\sqrt{2})W - \frac{40}{(1+\sqrt{2})W^2} = 0$$

$$W = 0.95 \text{ m}$$

$$H = 2.29 \text{ m}$$



Problem 2 - If only 1/2 of the solar cells receive light at any one time, and the power they can deliver is 100 watts per square meter, what is the power that this satellite can provide to the experiments and operating systems?

Surface area of the satellite:

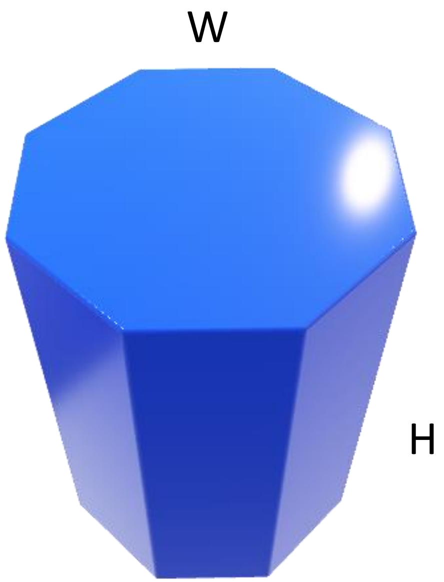
$$S = 4(1+\sqrt{2})W^2 + 8WH = 26.15\ \mathrm{m}^2$$

Only half of the solar cells can receive light at any given time.

$$\frac{1}{2}S = 13.08\ \mathrm{m}^2$$

Power delivered is 100 watts per square meter:

$$P = 100\ \mathrm{W/m}^2 \cdot 13.08\ \mathrm{m}^2 = 1308\ \text{watts}$$



Summary and Conclusions

- We analyzed and studied the velocity and acceleration profiles of STS-121 during its ascent to the ISS.
- We used optimization methods to design an optimal solar panel geometry for a satellite by minimizing the surface area.
- This research provides novel applications of the fundamental theorems of calculus to study motion in outer space and involves mathematical modeling, optimization, curve fitting, data analysis, and data visualization.

References

NASA Space Math
https://www.nasa.gov/stem-ed-resources/space-math-I.html

Ascent Data:
https://www.nasa.gov/pdf/468291main_STS-121_Ascent_Data.pdf